DIS 9B

1 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) What is the distribution of X?

(b) What is $\mathbb{P}(X = 7)$?

(c) What is $\mathbb{P}(X \geq 1)$?

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

2 Numbered Balls

Suppose you have a bag containing seven balls numbered $0, 1, 1, 2, 3, 5, 8$.

(a) You perform the following experiment: pull out a single ball, record its number, and replace it in the bag. If you repeat this experiment many times, what is the long-term average of the numbers that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls at a time and record their total. What is the long-term average of the numbers that you record?

3 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

(c) A building has n floors numbered $1, 2, \ldots, n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each gets off at a uniformly random one of the n floors (independently of everybody else). What is the expected number of floors the elevator stops at (not counting the ground floor)?

(d) A coin with heads probability p is flipped n times. A "run" is a maximal sequence of consecutive flips that are all the same. (Thus, for example, the sequence $HTHHHTTH$ with $n = 8$ has five runs.) Show that the expected number of runs is $1 + 2(n-1)p(1-p)$. Justify your calculation carefully.