CS 70 Discrete Mathematics and Probability Theory
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DIS 10A

1 Coupon Collection

Suppose you take a deck of n cards and repeatedly perform the following step: take the current top card and put it back in the deck at a uniformly random position (the probability that the card is placed in any of the n possible positions in the deck — including back on top — is $1/n$).

Consider the card that starts off on the bottom of the deck. What is the expected number of steps until this card rises to the top of the deck?

[*Hint*: Let T be the number of steps until the card rises to the top. We have $T = T_n + T_{n-1} + \cdots + T_2$, where the random variable T_i is the number of steps until the bottom card rises from position i to position $i-1$. Thus, for example, T_n is the number of steps until the bottom card rises off the bottom of the deck, and T_2 is the number of steps until the bottom card rises from second position to top position. What is the distribution of T_i?]

2 Rolling Dice

(a) If we roll a fair 6-sided die, what is the expected number of times we have to roll before we roll a 6? What is the variance?

(b) Suppose we have two fair n-sided dice labeled Die 1 and Die 2. If we roll the two dice until the value on Die 1 is smaller than the value on Die 2, what is the expected number of times that we roll? What is the variance?

3 Unreliable Servers

In a single cluster of a Google competitor, there are a huge number of servers, each with a uniform and independent probability of going down in a given day. On average, 4 servers go down in the cluster per day.

(a) What is an appropriate distribution by which the number of servers that crash can be modeled?

(b) Compute the expected value and variance of the number of crashed servers for a certain cluster.

(c) Compute the probability that less than 3 servers crashed.

(d) Compute the probability at least 3 servers crashed.

4 Will I Get My Package?

A delivery guy in some company is out delivering n packages to n customers, where $n \in \mathbb{N}$, $n > 1$. Not only does he hand a random package to each customer, he opens the package before delivering it with probability $1/2$. Let X be the number of customers who receive their own packages unopened.

(a) Compute the expectation $\mathbb{E}(X)$.

(b) Compute the variance $\text{var}(X)$.