

## HOMWORK 7

### Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

*I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.*

### 1 More Countability

Given:

- $A$  is a countable, non-empty set. For all  $i \in A$ ,  $S_i$  is an uncountable set.
- $B$  is an uncountable set. For all  $i \in B$ ,  $Q_i$  is a countable set.

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a)  $A \cap B$

(b)  $A \cup B$

(c)  $\bigcup_{i \in A} S_i$

(d)  $\bigcap_{i \in A} S_i$

(e)  $\bigcup_{i \in B} Q_i$

(f)  $\bigcap_{i \in B} Q_i$

## 2 Interval Bijection Construction

(a) Construct a bijective function  $f : [0, 1] \rightarrow (0, 1]$ .

(b) Construct a bijective function  $g : [0, 1] \rightarrow (0, 1)$ .

## 3 Counting Cartesian Products

For two sets  $A$  and  $B$ , define the Cartesian Product as  $A \times B = \{(a, b) : a \in A, b \in B\}$ .

(a) Given two countable sets  $A$  and  $B$ , prove that  $A \times B$  is countable.

(b) Given a finite number of countable sets  $A_1, A_2, \dots, A_n$ , prove that  $A_1 \times A_2 \times \dots \times A_n$  is countable.

(c) Consider an infinite number of countable sets:  $B_1, B_2, \dots$ . Under what condition(s) is  $B_1 \times B_2 \times \dots$  countable? Prove that if this condition is violated,  $B_1 \times B_2 \times \dots$  is uncountable.

## 4 Printing All $x$ where $M(x)$ Halts

(a) Prove that it is possible to write a program  $P$  which:

- takes as input  $M$ , a Java program,
- runs forever, and prints out strings to the console,
- for every  $x$ , if  $M(x)$  halts, then  $P(M)$  eventually prints out  $x$ ,
- for every  $x$ , if  $M(x)$  does NOT halt, then  $P(M)$  never prints out  $x$ .

(b) Lexicographical ordering of strings means (1) shorter strings are in front of longer strings and (2) for two strings of the same length, they are sorted in alphabetical order.

Prove that it's impossible to solve the above problem if we require the output be in lexicographical order.

## 5 Impossible Programs

Show that none of the following programs can exist.

(a) Consider a program  $P$  that takes in any program  $F$ , input  $x$  and output  $y$  and returns true if  $F(x)$  outputs  $y$  and returns false otherwise.

- (b) Consider a program  $P$  that takes in any program  $F$  and returns true if  $F(F)$  halts and returns false if it doesn't halt.
- (c) Consider a program  $P$  that takes in any programs  $F$  and  $G$  and returns true if  $F$  and  $G$  halt on all the same inputs and returns false otherwise.

## 6 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

- (a) How many ways are there to arrange  $n$  1s and  $k$  0s into a sequence?
- (b) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.  
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?
- (c) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?
- (d) How many 99-bit strings are there that contain more ones than zeros?
- (e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.  
How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?
- (f) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E.
- (g) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (h) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).
- (i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (j) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

- (k) How many solutions does  $x_0 + x_1 + \cdots + x_k = n$  have, if each  $x$  must be a non-negative integer?
- (l) How many solutions does  $x_0 + x_1 = n$  have, if each  $x$  must be a *strictly positive* integer?
- (m) How many solutions does  $x_0 + x_1 + \cdots + x_k = n$  have, if each  $x$  must be a *strictly positive* integer?