CS 70 Discrete Mathematics and Probability Theory
Fall 2017 Kannan Ramchandran and Satish Rao

HW 11

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Proof with Indicators

Let $n \in \mathbb{Z}_+$. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and let $A_1, \dots, A_n$ be events. Prove that $\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \mathbb{P}(A_i \cap A_j) \geq 0$.

2 Balls and Bins

Throw n balls into m bins, where m and n are positive integers. Let X be the number of bins with exactly one ball. Compute $\operatorname{var} X$.

3 Portfolio Optimization

Suppose that there are n assets, where n is a positive integer. For each unit dollar invested in asset i, for $i = 1, \dots, n$, with probability p_i the value of the asset will grow by α_i to $1 + \alpha_i$, and with probability $1 - p_i$ the value of the asset will shrink by α_i to $1 - \alpha_i$. Let the proportion of money invested in asset i be w_i (so that $\sum_{i=1}^{n} w_i = 1$), and let X_i be a random variable denoting the final value of the ith asset per unit dollar. Then $X = w_1X_1 + \cdots + w_nX_n$ is the total value. For simplicity, assume that the outcomes of the different assets are independent.

(a) Compute the expectation $\mathbb{E}[X]$. What values of w_i maximize this quantity?

(b) Compute the variance $\operatorname{var} X$. What values of w_i minimize this quantity?

4 Uniform Means

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed uniform random variables on the interval $[0,1]$ (where n is a positive integer).

(a) Let $Y = \min\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Y)$. [*Hint*: Use the tail sum formula, which says the expected value of a nonnegative random variable is $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$. Note that we can use the tail sum formula since $Y \geq 0$.]

(b) Let $Z = \max\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Z)$. [*Hint*: Find the CDF.]

5 Darts (Again!)

Alvin is playing darts. His aim follows an exponential distribution; that is, the probability density that the dart is x distance from the center is $f_X(x) = \exp(-x)$. The board's radius is 4 units.

(a) What is the probability the dart will stay within the board?

(b) Say you know Alvin made it on the board. What is the probability he is within 1 unit from the center?

(c) If Alvin is within 1 unit from the center, he scores 4 points, if he is within 2 units, he scores 3, etc. In other words, Alvin scores $\lfloor 5 - x \rfloor$, where x is the distance from the center. What is Alvin's expected score after one throw?

6 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probablity that the new bulb will last at least 30 days?

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?