70: Discrete Math and Probability Theory

Course Webpage: **http://www.eecs70.org/**

Explains policies, has office hours, homework, midterm dates, etc.

Two midterms, final.
Dates on website.

Some options on how to be graded. Also on website.

Will give guidance and data on each approach.

Questions/Announcements $\Longrightarrow$ piazza:
piazza.com/berkeley/fall2017/cs70

70: Discrete Math and Probability Theory

Programming + Microprocessors $\equiv$ Superpower!

What are your super powerful programs/processors doing?
Logic and Proofs!
Induction $\equiv$ Recursion.

What can computers do?
Work with discrete objects.
Discrete Math $\implies$ immense application.

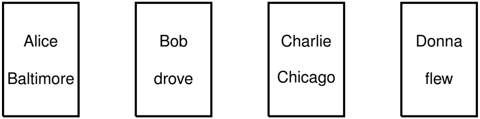
Computers learn and interact with the world?
E.g. machine learning, data analysis, robotics, ...
Probability!

See note 1, for more discussion.

Wason's experiment:1

Suppose we have four cards on a table:

- 1st about Alice, 2nd about Bob, 3rd Charlie, 4th Donna.
- Card contains person's destination on one side,
  and mode of travel.
- Consider the theory:
  "If a person travels to Chicago, he/she flies."
- Suppose you see that Alice went to Baltimore, Bob drove,
  Charlie went to Chicago, and Donna flew.



- Which cards must you flip to test the theory?

Answer: Later.

CS70: Lecture 1. Outline.

Today: Note 1.   Note 0 is background. Do read it.

The language of proofs!

1. Propositions.
2. Propositional Forms.
3. Implication.
4. Truth Tables
5. Quantifiers
6. More De Morgan's Laws

Propositions: Statements that are true or false.

$\sqrt{2}$ is irrational	**Proposition**	True
2+2 = 4	**Proposition**	True
2+2 = 3	**Proposition**	False
826th digit of pi is 4	**Proposition**	False
Johny Depp is a good actor	**Not a Proposition**	
All evens >2 are sums of 2 primes	**Proposition**	False
$4+5$	**Not a Proposition.**	
$x+x$	**Not a Proposition.**	
Alice travelled to Chicago	**Proposition.**	False

Again: “value” of a proposition is ... True or False

Propositional Forms.

Put propositions together to make another...

Conjunction ("and"): $P \wedge Q$

"$P \wedge Q$" is True when both P and Q are True . Else False .

Disjunction ("or"): $P \vee Q$

"$P \vee Q$" is True when at least one P or Q is True . Else False .

Negation ("not"): $\neg P$

"$\neg P$" is True when P is False . Else False .

Examples:

$\neg$ "$(2+2=4)$" – a proposition that is ... False

"$2+2=3$" $\bigwedge$ "$2+2=4$" – a proposition that is ... False

"$2+2=3$" $\bigvee$ "$2+2=4$" – a proposition that is ... True

Propositional Forms: quick check!

$P =$ "$\sqrt{2}$ is rational"
$Q =$ "826th digit of pi is 2"

P is ...False .
Q is ...True .

$P \wedge Q$ ... False

$P \vee Q$ ... True

$\neg P$ ... True

Put them together..

Propositions:

P_1 - Person 1 rides the bus.
P_2 - Person 2 rides the bus.
....

But we can't have either of the following happen; That either person 1 or person 2 ride the bus and person 3 or 4 ride the bus. Or that person 2 or person 3 ride the bus and that either person 4 rides the bus or person 5 doesn't.

Propositional Form:

$\neg(((P_1 \vee P_2) \wedge (P_3 \vee P_4)) \vee ((P_2 \vee P_3) \wedge (P_4 \vee \neg P_5)))$

Can person 3 ride the bus?
Can person 3 and person 4 ride the bus together?

This seems ...complicated.

We can program!!!!

We need a way to keep track!

Truth Tables for Propositional Forms.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Notice: $\wedge$ and $\vee$ are commutative.

One use for truth tables: Logical Equivalence of propositional forms!

Example: $\neg(P \wedge Q)$ logically equivalent to $\neg P \vee \neg Q$

...because the two propositional forms have the same...

....Truth Table!

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

DeMorgan's Law's for Negation: distribute and flip!

$\neg(P \wedge Q) \quad \equiv \quad \neg P \vee \neg Q$ $\qquad \neg(P \vee Q) \quad \equiv \quad \neg P \wedge \neg Q$

Distributive?

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$?

Simplify: $(T \wedge Q) \equiv Q$, $(F \wedge Q) \equiv F$.

Cases:

P is True .

LHS: $T \wedge (Q \vee R) \equiv (Q \vee R)$.

RHS: $(T \wedge Q) \vee (T \wedge R) \equiv (Q \vee R)$.

P is False .

LHS: $F \wedge (Q \vee R) \equiv F$.

RHS: $(F \wedge Q) \vee (F \wedge R) \equiv (F \vee F) \equiv F$.

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$?

Simplify: $T \vee Q \equiv T$, $F \vee Q \equiv Q$.

Foil 1:

$(A \vee B) \wedge (C \vee D) \equiv (A \wedge C) \vee (A \wedge D) \vee (B \wedge C) \vee (B \wedge D)$?

Foil 2:

$(A \wedge B) \vee (C \wedge D) \equiv (A \vee C) \wedge (A \vee D) \wedge (B \vee C) \wedge (B \vee D)$?

Implication.

$P \implies Q$ interpreted as

If P, then Q.

True Statements: P, $P \implies Q$.
Conclude: Q is true.

Examples:

Statement: If you stand in the rain, then you'll get wet.
P = "you stand in the rain"
Q = "you will get wet"
Statement: "Stand in the rain"
Can conclude: "you'll get wet."

Statement:
If right triangle has sidelengths $a, b \leq c$, then $a^2 + b^2 = c^2$.

P = "a right triangle has sidelengths $a \leq b \leq c$",
Q = "$a^2 + b^2 = c^2$".

Non-Consequences/consequences of Implication

The statement "$P \implies Q$"

only is False if P is True and Q is False .

False implies nothing
P False means Q can be True or False
Anything implies true.
P can be True or False when Q is True

If chemical plant pollutes river, fish die.
If fish die, did chemical plant pollute river?

Not necessarily.

$P \implies Q$ and Q are True does not mean P is True

Be careful!

Instead we have:
$P \implies Q$ and P are True *does* mean Q is True .

The chemical plant pollutes river. Can we conclude fish die?

Some Fun: use propositional formulas to describe implication?
$((P \implies Q) \wedge P) \implies Q$.

Implication and English.

$P \implies Q$

- If P, then Q.
- Q if P.
  Just reversing the order.
- P only if Q.
  Remember if P is true then Q must be true.
  this suggests that P can only be true if Q is true.
  since if Q is false P must have been false.
- P is sufficient for Q.
  This means that proving P allows you
  to conclude that Q is true.
- Q is necessary for P.
  For P to be true it is necessary that Q is true.
  Or if Q is false then we know that P is false.

Truth Table: implication.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg P \vee Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg P \vee Q \quad \equiv P \implies Q$.

These two propositional forms are logically equivalent!

Contrapositive, Converse

- Contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$.
  - If the plant pollutes, fish die.
  - If the fish don't die, the plant does not pollute. (contrapositive)
  - If you stand in the rain, you get wet.
  - If you did not stand in the rain, you did not get wet. (not contrapositive!) converse!
  - If you did not get wet, you did not stand in the rain. (contrapositive.)

  Logically equivalent! Notation: $\equiv$.
  $P \implies Q \equiv \neg P \vee Q \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \implies \neg P$.
- Converse of $P \implies Q$ is $Q \implies P$.

  If fish die the plant pollutes.

  Not logically equivalent!
- **Definition:** If $P \implies Q$ and $Q \implies P$ is P if and only if Q or $P \iff Q$.
  (Logically Equivalent: $\iff$.)

Variables.

Propositions?

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- $x > 2$
- n is even and the sum of two primes

No. They have a free variable.

We call them predicates, e.g., $Q(x) =$ "x is even"
Same as boolean valued functions from 61A or 61AS!

- $P(n) =$ "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$."
- $R(x) =$ "$x > 2$"
- $G(n) =$ "n is even and the sum of two primes"
- Remember Wason's experiment!
  $F(x) =$ "Person x flew."
  $C(x) =$ "Person x went to Chicago
- $C(x) \implies F(x)$. Theory from Wason's.
  If person x goes to Chicago then person x flew.

Next: Statements about boolean valued functions!!

Quantifiers..

There exists quantifier:

$(\exists x \in S)(P(x))$ means "There exists an x in S where $P(x)$ is true."

For example:

$$(\exists x \in \mathbb{N})(x = x^2)$$

Equivalent to "$(0 = 0) \vee (1 = 1) \vee (2 = 4) \vee \ldots$"

Much shorter to use a quantifier!

For all quantifier;
$(\forall x \in S)\ (P(x))$. means "For all x in S, we have $P(x)$ is True ."

Examples:

"Adding 1 makes a bigger number."

$$(\forall x \in \mathbb{N})\ (x + 1 > x)$$

"the square of a number is always non-negative"

$$(\forall x \in \mathbb{N})(x^2 >= 0)$$

Wait! What is $\mathbb{N}$?

Quantifiers: universes.

Proposition: "For all natural numbers n**,** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$**."**

Proposition has **universe**: "the natural numbers".

Universe examples include..

- $\mathbb{N} = \{0, 1, \ldots\}$ (natural numbers).
- $\mathbb{Z} = \{\ldots, -1, 0, \ldots\}$ (integers)
- $\mathbb{Z}^+$ (positive integers)
- $\mathbb{R}$ (real numbers)
- Any set: $S = \{Alice, Bob, Charlie, Donna\}$.
- See note 0 for more!

Lecture ended here. Will continue from here on Friday.