CS70: Lecture 11. Outline.

1. Public Key Cryptography
2. RSA system
   2.1 Efficiency: Repeated Squaring.
   2.2 Correctness: Fermat's Theorem.
   2.3 Construction.
3. Warnings.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

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$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Lots of Mods

$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let's try 5.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!
Let's try 3.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!
Let's try 3. Not $5 \pmod 7$!

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!
Let's try 3. Not $5 \pmod 7$!

Lots of Mods

$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod{5}$!
Let's try 3. Not $5 \pmod{7}$!

If $x = 6 \pmod{7}$

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!
Let's try 3. Not $5 \pmod 7$!

If $x = 6 \pmod 7$
then x is in $\{5, 12, 19, 26, 33\}$.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

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Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

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If $x = 6 \pmod 7$
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Oh, only 33 is $3 \pmod 5$.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

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Let's try 3. Not $5 \pmod 7$!

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Oh, only 33 is $3 \pmod 5$.

Hmmm...

Lots of Mods

$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

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Let's try 3. Not $5 \pmod{7}$!

If $x = 6 \pmod{7}$
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Oh, only 33 is $3 \pmod{5}$.

Hmmm... only one solution.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

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Let's try 3. Not $5 \pmod 7$!

If $x = 6 \pmod 7$
then x is in $\{5, 12, 19, 26, 33\}$.

Oh, only 33 is $3 \pmod 5$.

Hmmm... only one solution.

A bit slow for large values.

Simple Chinese Remainder Theorem.

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My love is won.

Simple Chinese Remainder Theorem.

My love is won. Zero and One.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

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Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

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CRT Thm: Unique solution $\pmod{mn}$.

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Proof:

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

CRT Thm: Unique solution $\pmod{mn}$.
Proof:
Consider $u = n(n^{-1} \pmod{m})$.

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$u = 0 \pmod{n}$

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$u = 0 \pmod{n}$ $\quad$ $u = 1 \pmod{m}$

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$u = 0 \pmod{n}$ $\qquad u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.

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My love is won. Zero and One. Nothing and nothing done.

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Let $x = au + bv$.

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$v = 1 \pmod{n}$ $\quad$ $v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$

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$v = 1 \pmod{n}$ $\qquad v = 0 \pmod{m}$

Let $x = au + bv$.

$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$

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$v = 1 \pmod{n}$ $\qquad v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution?

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Only solution? If not, two solutions, x and y.

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$v = 1 \pmod{n}$ $\quad$ $v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.
$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m,n)=1.

CRT Thm: Unique solution $\pmod{mn}$.
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$u = 0 \pmod{n}$ $\quad$ $u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.
$v = 1 \pmod{n}$ $\quad$ $v = 0 \pmod{m}$

Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.

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My love is won. Zero and One. Nothing and nothing done.

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn$

Simple Chinese Remainder Theorem.

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn \implies x,y \notin \{0,\ldots,mn-1\}$.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m,n)=1.

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$\implies x-y \geq mn \implies x,y \notin \{0,\ldots,mn-1\}$.
Thus, only one solution modulo mn.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

CRT Thm: Unique solution $\pmod{mn}$.
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Let $x = au + bv$.
$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
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Thus, only one solution modulo mn. $\square$

Xor

Computer Science:

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Computer Science:
1 - True
0 - False

Xor

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$1 \vee 1 = 1$

Xor

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Xor

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1 - True
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$A \oplus B$ - Exclusive or.

Xor

Computer Science:
1 - True
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  1 - True
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$A \oplus B$ - Exclusive or.
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Note: Also modular addition modulo 2!

Xor

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 1 - True
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Note: Also modular addition modulo 2!
 $\{0, 1\}$ is set. Take remainder for 2.

Xor

Computer Science:
1 - True
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$0 \vee 0 = 0$

Note: Also modular addition modulo 2!

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Property: $A \oplus B \oplus B = A$.

Xor

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$A \oplus B$ - Exclusive or.
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Note: Also modular addition modulo 2!
$\{0, 1\}$ is set. Take remainder for 2.

Property: $A \oplus B \oplus B = A$.
By cases: $1 \oplus 1 \oplus 1 = 1$.

Xor

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  1 - True
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Property: $A \oplus B \oplus B = A$.
By cases: $1 \oplus 1 \oplus 1 = 1$. ...

Cryptography ...

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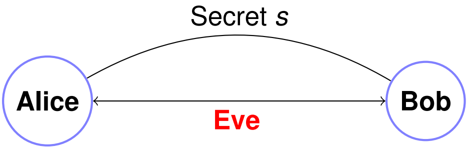
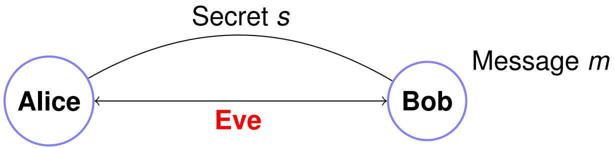
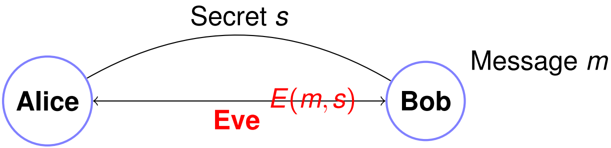


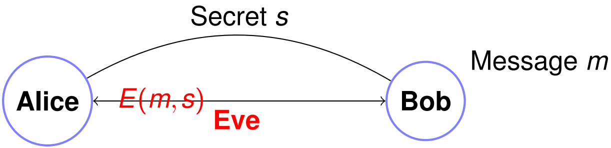
Cryptography ...



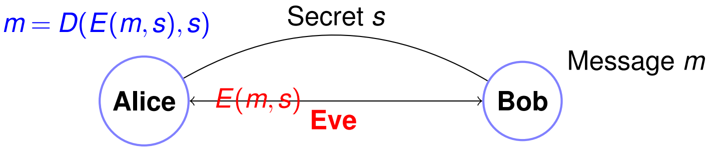
Cryptography ...



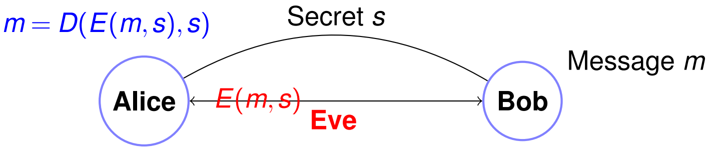
Cryptography ...



Cryptography ...

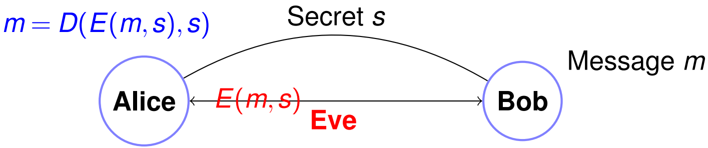


Cryptography ...



Example:

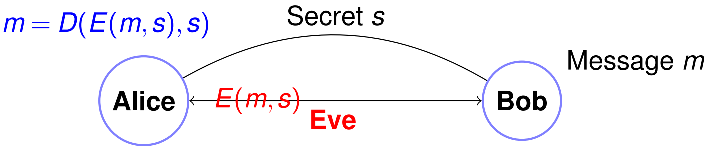
Cryptography ...



Example:

One-time Pad: secret s is string of length $|m|$.

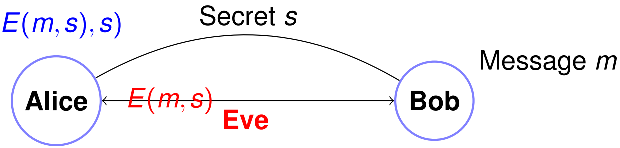
Cryptography ...



Example:
One-time Pad: secret s is string of length $|m|$.
$m = 1010101110101101$

Cryptography ...

$m = D(E(m, s), s)$



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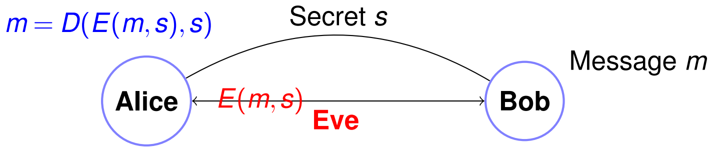

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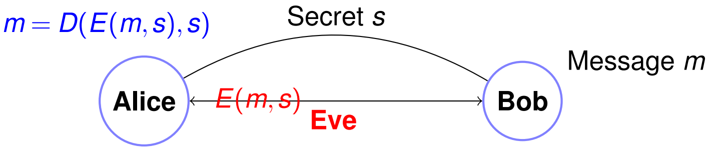

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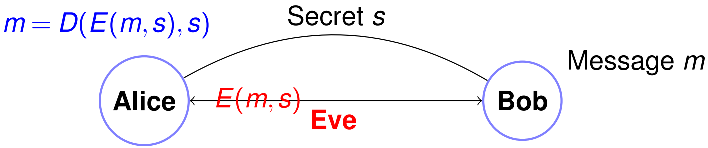

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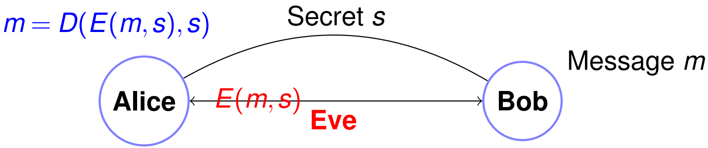

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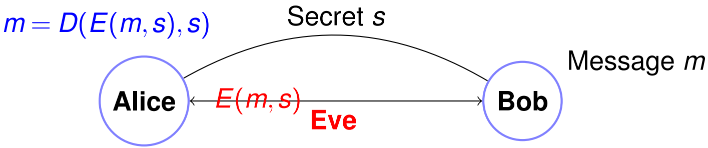

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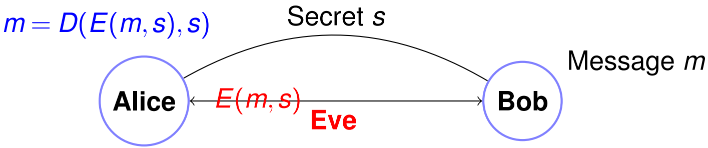

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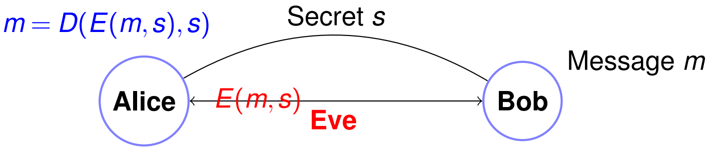

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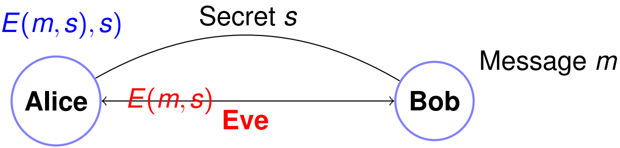

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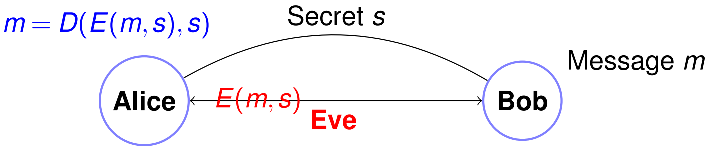

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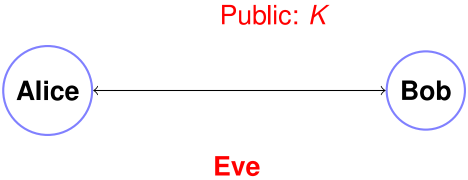

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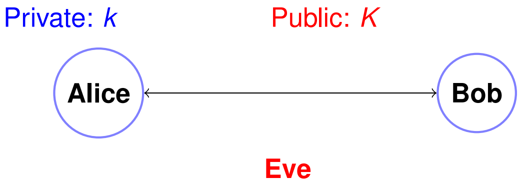

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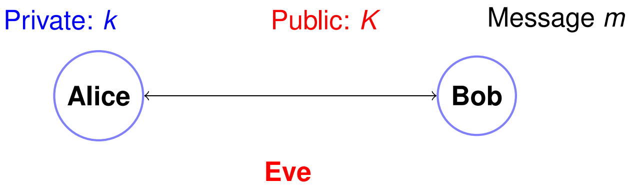

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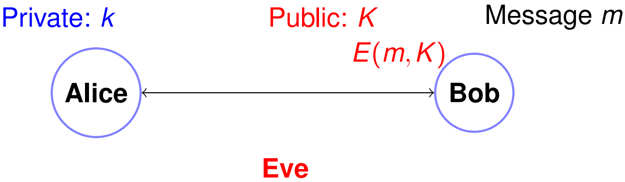

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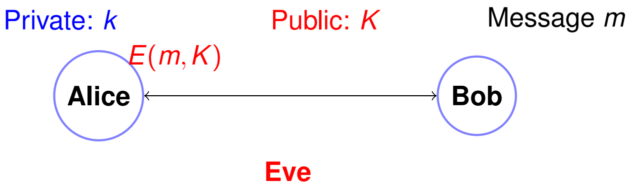

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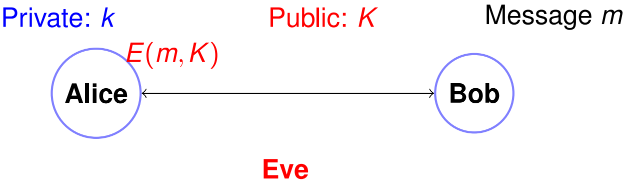

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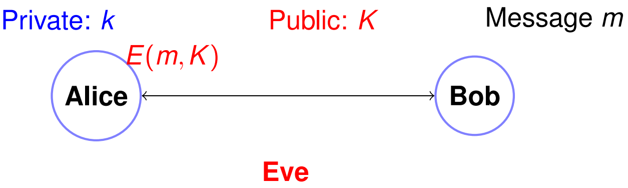

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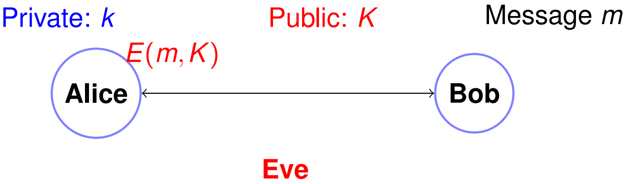

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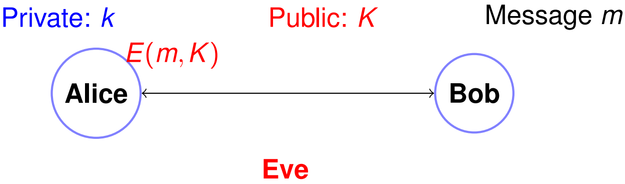

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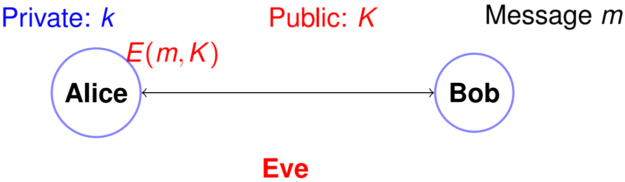

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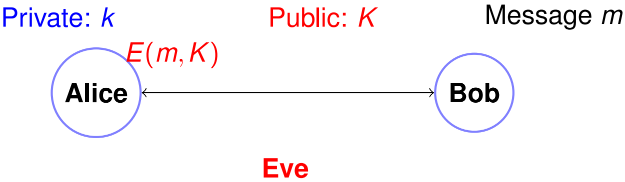

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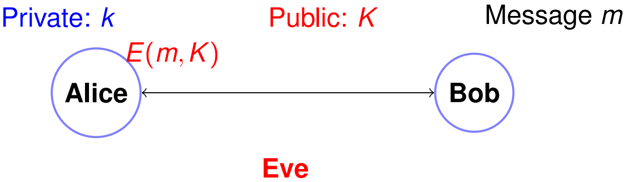

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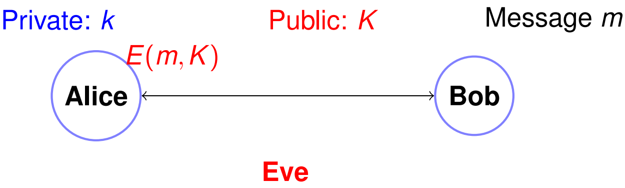

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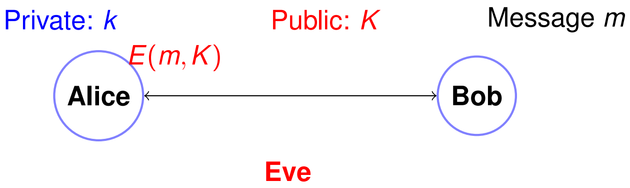

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Is this even possible?

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$d = e^{-1} = -17 = 43 = \pmod{60}$

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In general, $O(N)$ or $O(2^n)$ multiplications!

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4 multiplications sort of...
Need to compute $51^{32} \ldots 51^{1}$.?

Repeated squaring.

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$51^{1} \equiv 51 \pmod{77}$

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Need to compute $51^{32} \ldots 51^{1}$.?
$51^1 \equiv 51 \pmod{77}$
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5 more multiplications.

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Decoding got the message back!

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Decoding got the message back!

Repeated Squaring took 9 multiplications

Repeated squaring.

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$51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} = (60)*(53)*(60)*(51) \equiv 2 \pmod{77}$.

Decoding got the message back!

Repeated Squaring took 9 multiplications versus 43.

Repeated Squaring: x^y

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Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute x^1,

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2,$

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4,$

Repeated Squaring: x^y

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Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.
2. Multiply together x^i where the $(\log(i))$th bit of y (in binary) is 1.

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

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   Example:

Repeated Squaring: x^y

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   Example: $43 = 101011$ in binary.

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Repeated Squaring: x^y

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Modular Exponentiation: $x^y \mod N$.

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

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Modular Exponentiation: $x^y \mod N$. All n-bit numbers. Repeated Squaring:

Repeated Squaring: x^y

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Repeated Squaring: x^y

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Modular Exponentiation: $x^y \mod N$. All n-bit numbers. Repeated Squaring:

$O(n)$ multiplications.
$O(n^2)$ time per multiplication.

Repeated Squaring: x^y

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Modular Exponentiation: $x^y \mod N$. All n-bit numbers. Repeated Squaring:

$O(n)$ multiplications.
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$\implies O(n^3)$ time.

Conclusion: $x^y \mod N$

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

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Modular Exponentiation: $x^y \mod N$. All n-bit numbers. Repeated Squaring:

$O(n)$ multiplications.
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Conclusion: $x^y \mod N$ takes $O(n^3)$ time.

RSA is pretty fast.

Modular Exponentiation: $x^y \mod N$.

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Remember RSA encoding/decoding!

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Remember RSA encoding/decoding!

$E(m,(N,e)) = m^e \pmod{N}$.

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Always decode correctly?

$$E(m,(N,e)) = m^e \pmod{N}.$$

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$d = e^{-1} \pmod{(p-1)(q-1)} \iff ed = k(p-1)(q-1)+1$.

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$E(m,(N,e)) = m^e \pmod N$.
$D(m,(N,d)) = m^d \pmod N$.

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Another view:
$d = e^{-1} \pmod{(p-1)(q-1)} \iff ed = k(p-1)(q-1)+1$.

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