Today.

Polynomials.

Secret Sharing.

Erasure Coding.

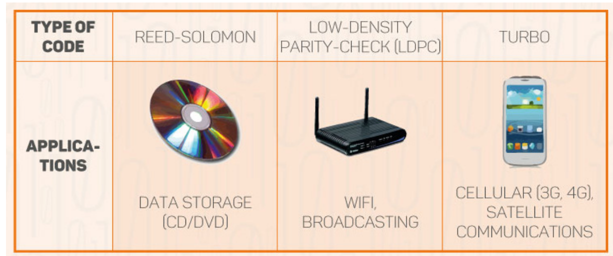
Secret Sharing.

Share secret among n **people.**

Secrecy: Any $k-1$ knows nothing.
Robustness: Any k knows secret.
Efficient: Minimize storage.

Illustration: need at least 3 keys to open a bank vault

Other apps. we'll see: codes based on polynomials



TYPE OF CODE	REED-SOLOMON	LOW-DENSITY PARITY-CHECK (LDPC)	TURBO
APPLICA-TIONS	DATA STORAGE (CD/DVD)	WIFI, BROADCASTING	CELLULAR (3G, 4G), SATELLITE COMMUNICATIONS

Back to secret sharing: idea of the day

Two points make a line.

Lots of lines go through one point.

Secret message is represented as the y-intercept.

Let's recall how polynomials work.

Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1}x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

$P(x)$ **contains** point (a,b) if $b = P(a)$.

Polynomials over reals: $a_1, \ldots, a_d \in \Re$, use $x \in \Re$.

Polynomials $P(x)$ **with arithmetic modulo** p: [1] $a_i \in \{0, \ldots, p-1\}$ and

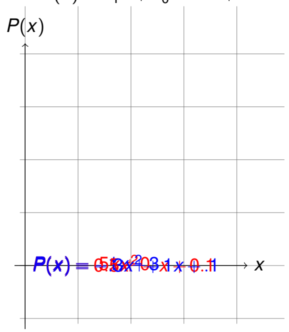
$$P(x) = a_d x^d + a_{d-1}x^{d-1} \cdots + a_0 \pmod p,$$

for $x \in \{0, \ldots, p-1\}$.

[1] A field is a set of elements with addition and multiplication operations, with inverses. $GF(p) = (\{0, \ldots, p-1\}, + \pmod p, * \pmod p)$.

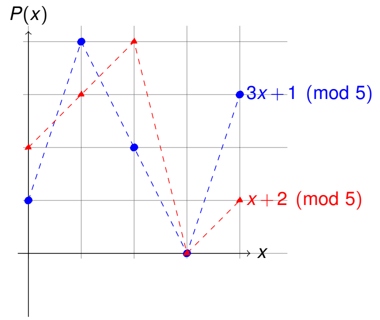
Polynomial: $P(x) = a_d x^4 + \cdots + a_0$

Line: $P(x) = a_1 x + a_0 = mx + b$



Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$

Polynomial: $P(x) = a_d x^4 + \cdots + a_0 \pmod{p}$



Finding an intersection.
$x+2 \equiv 3x+1 \pmod 5$
$\implies 2x \equiv 1 \pmod 5 \implies x \equiv 3 \pmod 5$
3 is multiplicative inverse of 2 modulo 5.
Good when modulus is prime!!

Two points make a line.

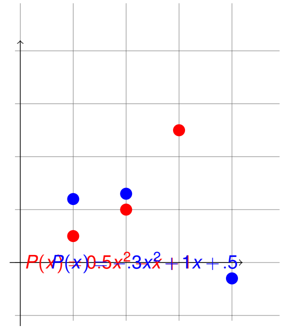
Fact: There is exactly 1 polynomial having degree $\leq d$ containing $d+1$ points. [2]

Two points specify a line. Three points specify a parabola.

Modular Arithmetic Fact: There is exactly 1 polynomial having degree $\leq d$ (with arithmetic modulo prime p) containing $d+1$ pts.

[2]Points with different x values.

3 points determine a parabola.

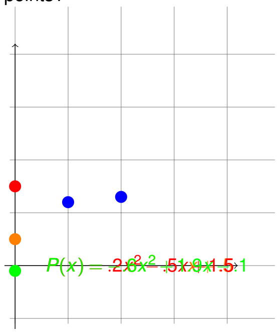


Fact: Exactly 1 polynomial having degree$\leq d$ polynomial contains $d+1$ points. [3]

[3]Points with different x values.

2 points not enough.

Question: How many parabolas exist that contain exactly 2 distinct points?



A parabola $P(x)$ containing 2 blue points can contain *any* $(0, y)$!

Modular Arithmetic Fact and Secrets

Modular Arithmetic Fact: Exactly 1 polynomial having degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Shamir's k out of n Scheme:
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and random $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Robustness: Any k shares gives secret.
Knowing k pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts $\implies$ any $P(0)$ is possible.

From $d+1$ points to degree d polynomial?

For a line, $a_1x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

$$P(1) = m(1) + b \equiv m + b \equiv 3 \pmod 5$$
$$P(2) = m(2) + b \equiv 2m + b \equiv 4 \pmod 5$$

Subtract first from second..

$$m + b \equiv 3 \pmod 5$$
$$m \equiv 1 \pmod 5$$

Backsolve: $b \equiv 2 \pmod 5$. Secret is 2.

And the line is...

$$x + 2 \mod 5.$$

Quadratic

For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.
Plug in points to find equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 5 \\ P(3) = 4a_2 + 3a_1 + a_0 &\equiv 0 \pmod 5 \end{aligned}$$

$$\begin{aligned} a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ 3a_1 + 2a_0 &\equiv 1 \pmod 5 \\ 4a_1 + 2a_0 &\equiv 2 \pmod 5 \end{aligned}$$

Subtracting 2nd from 3rd yields: $a_1 = 1$.
$a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1}) = (3)(3) = 9 \equiv 4 \pmod 5$
$a_2 = 2 - 1 - 4 \equiv 2 \pmod 5$.

So polynomial is $2x^2 + 1x + 4 \pmod 5$

In general..

Given points: $(x_1,y_1);(x_2,y_2)\cdots(x_k,y_k)$.

Solve...

$$\begin{aligned} a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod p \\ a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod p \\ &\cdot \\ &\cdot \\ a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod p \end{aligned}$$

Will this always work?

As long as solution **exists** and it is **unique!** And...

Modular Arithmetic Fact: Exactly 1 polynomial having degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Another Construction: Lagrange Interpolation!

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3);(2,4);(3,0)$.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

Try $(x-2)(x-3) \pmod 5$.

Value is 0 at 2 and 3. Value is 2 at 1. Not 1! Doh!!
So "Divide by 2" or multiply by 3.
$\Delta_1(x) = (x-2)(x-3)(3) \pmod 5$ contains $(1,1);(2,0);(3,0)$.

$\Delta_2(x) = (x-1)(x-3)(4) \pmod 5$ contains $(1,0);(2,1);(3,0)$.

$\Delta_3(x) = (x-1)(x-2)(3) \pmod 5$ contains $(1,0);(2,0);(3,1)$.

But wanted to hit $(1,3);(2,4);(3,0)$!

$P(x) = 3\Delta_1(x) + 4\Delta_2(x) + 0\Delta_3(x)$ works.

Same as before?

...after a lot of calculations... $P(x) = 2x^2 + 1x + 4 \mod 5$.

The same as before!

Polynomials.

We will work with polynomials with arithmetic modulo p.

Everything has a multiplicative inverse.

Like rationals, reals.

Warning: no calculus, and no order even.

Delta Polynomials: Concept.

For set of x-values, $x_1,\ldots,x_{d+1}$.

$$\Delta_i(x) = \begin{cases} 1, & \text{if } x = x_i. \\ 0, & \text{if } x = x_j \text{ for } j \neq i. \\ ?, & \text{otherwise.} \end{cases} \quad (1)$$

Given $d+1$ points, use Δ_i functions to go through points? $(x_1,y_1),\ldots,(x_{d+1},y_{d+1})$.

Will $y_1\Delta_1(x)$ contain (x_1,y_1)?

Will $y_2\Delta_2(x)$ contain (x_2,y_2)?

Does $y_1\Delta_1(x) + y_2\Delta_2(x)$ contain (x_1,y_1)? and (x_2,y_2)?

See the idea? Function that contains all points?

$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x)\ldots + y_{d+1}\Delta_{d+1}(x)$.

There exists a polynomial...

Modular Arithmetic Fact: Exactly 1 polynomial having degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Proof of at least one polynomial:
Given points: $(x_1,y_1);(x_2,y_2)\cdots(x_{d+1},y_{d+1})$.

$$\Delta_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}.$$

Numerator is 0 at $x_j \neq x_i$.

Denominator makes it 1 at x_i.

And..

$$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) + \cdots + y_{d+1}\Delta_{d+1}(x).$$

hits points $(x_1,y_1);(x_2,y_2)\cdots(x_{d+1},y_{d+1})$. Degree d polynomial!

Construction proves the existence of a polynomial!

Example.

$\Delta_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}$.

Degree 1 polynomial, $P(x)$, that contains $(1,3)$ and $(3,4)$?

Work modulo 5.

$\Delta_1(x)$ contains $(1,1)$ and $(3,0)$.

$\Delta_1(x) = \frac{(x-3)}{1-3} = \frac{x-3}{-2}$
$= 2(x-3) = 2x-6 = 2x+4 \pmod 5$.

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3);(2,4);(3,0)$.

Work modulo 5.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

$\Delta_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2} = 3(x-2)(x-3)$
$= 3x^2+3 \pmod 5$

Put the delta functions together.

In general.

Given points: $(x_1,y_1);(x_2,y_2)\cdots(x_k,y_k)$.

$$\Delta_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)}.$$

Numerator is 0 at $x_j \neq x_i$.

Denominator makes it 1 at x_i.

And..

$$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) + \cdots + y_k\Delta_k(x).$$

hits points $(x_1,y_1);(x_2,y_2)\cdots(x_k,y_k)$.

Construction proves the existence of the polynomial!

Where have we seen this concept before? (Hint: CRT)

My love is won. Zero and one. Nothing done.

Find $x = a \pmod m$ and $x = b \pmod n$ where gcd(m,n)=1.

Solution: $x = au + bv$, where

$u = 0 \pmod n$ and $u = 1 \pmod m$
$v = 1 \pmod n$ and $v = 0 \pmod m$

Similar deal here with the Delta polynomials in Lagrange interpolation.

Uniqueness.

Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

Proof:

Roots fact: Any degree d polynomial has at most d roots.

Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$R(x) = Q(x) - P(x)$ has $d+1$ roots and is degree d.
Contradiction.

□

Must prove **Roots fact.**

Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x-3)$ modulo 5.

```
              4 x + 4 r 4
          -----------------
x - 3 )  4x^2 - 3 x + 2
         4x^2 - 2x
         ----------
                4x + 2
                4x - 2
                -------
                     4
```

$4x^2 - 3x + 2 \equiv (x-3)(4x+4) + 4 \pmod 5$

In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder r.

That is, $P(x) = (x-a)Q(x) + r$

Only d roots.

Lemma 1: $P(x)$ has root a iff $P(x)/(x-a)$ has remainder 0:
$P(x) = (x-a)Q(x)$.

Proof: $P(x) = (x-a)Q(x) + r$.
Plugin a: $P(a) = r$.
It is a root if and only if $r = 0$.

□

Lemma 2: $P(x)$ has d roots; $r_1,\ldots,r_d$ then
$P(x) = c(x-r_1)(x-r_2)\cdots(x-r_d)$.
Proof Sketch: By induction.

Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1. $Q(x)$ has smaller degree so use the induction hypothesis.

□

$d+1$ roots implies degree is at least $d+1$.

Roots fact: Any degree d polynomial has at most d roots.

Finite Fields

Proof works for reals, rationals, and complex numbers.

..but not for integers, since no multiplicative inverses.

Arithmetic modulo a prime p has multiplicative inverses..

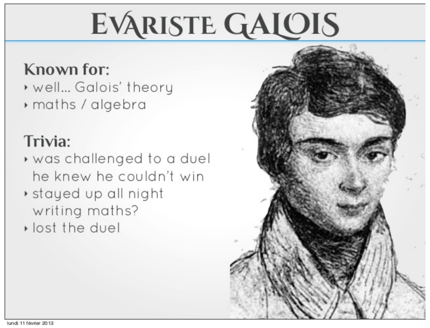
..and has only a finite number of elements.

Good for computer science.

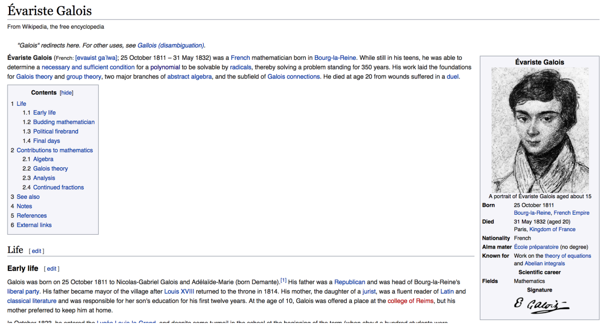
Arithmetic modulo a prime m is a **finite field**
denoted by F_m or $GF(m)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

History lesson: Evariste Galois (1811-1832)



History lesson: Evariste Galois (1811-1832)



Secret Sharing

Modular Arithmetic Fact: There exists exactly one polynomial having degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

Shamir's k out of n Scheme:
Secret $s \in \{0,\ldots,p-1\}$

1. Choose $a_0 = s$, and randomly $a_1,\ldots,a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Robustness: Any k knows secret.
Knowing k pts, only one $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ knows nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

Minimality.

Need $p > n$ to hand out n shares: $P(1)\ldots P(n)$.

For b-bit secret, must choose a prime $p > 2^b$.

Theorem: There is always a prime between n and $2n$.
Chebyshev said it,
And I say it again,
There is always a prime
Between n and 2n.

Working over numbers within 1 bit of secret size. **Minimality.**

With k shares, reconstruct polynomial, $P(x)$.

With $k-1$ shares, any of p values possible for $P(0)$!

(Almost) any b-bit string possible!

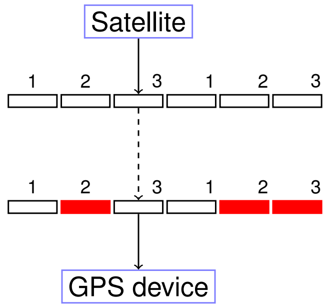
(Almost) the same as what is missing: one $P(i)$.

Runtime.

Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k-1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

Erasure Codes.

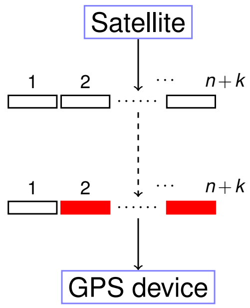


3 packet message. So send 6!

Lose 3 out 6 packets.

Gets packets 1,1,and 3.

Erasure Codes.



n packet message. So send $n+k$!

Lose k packets.

Any n packets is enough!

n packet message.

Optimal.

Solution Idea.

n packet message, channel that loses k packets.

Must send $n+k$ packets!

Any n packets should allow reconstruction of n packet message.

Any n point values allow reconstruction of degree $n-1$ polynomial.

Alright!!!!!!

Use polynomials.

Information Theory.

Size: Can choose a prime between 2^{b-1} and 2^b.
(Lose at most 1 bit per packet.)

But: packets need label for x value.

There are Galois Fields $GF(2^n)$ where one loses nothing.

– Can also run the Fast Fourier Transform.

In practice, $O(n)$ operations with almost the same redundancy.

Comparison with Secret Sharing: information content.

Secret Sharing: each share is size of whole secret.
Coding: Each packet has size $1/n$ of the whole message.

Erasure Codes.

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

A degree $n-1$ polynomial determined by any n points!

Erasure Coding Scheme: message = $m_0, m_2, \ldots, m_{n-1}$.

1. Choose prime $p \approx 2^b$ for packet size b.
2. $P(x) = m_{n-1}x^{n-1} + \cdots m_0 \pmod{p}$.
3. Send $P(1), \ldots, P(n+k)$.

Any n of the $n+k$ packets gives polynomial ...and message!

Erasure Code: Example.

Send message of 1,4, and 4.

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

How?

Lagrange Interpolation.
Linear System.

Work modulo 5.

$P(x) = x^2 \pmod 5$
$P(1) = 1, P(2) = 4, P(3) = 9 = 4 \pmod 5$

Send $(0, P(0)) \ldots (5, P(5))$.

6 points. Better work modulo 7 at least!

Why? $(0, P(0)) = (5, P(5)) \pmod 5$

Example

Make polynomial with $P(1)=1$, $P(2)=4$, $P(3)=4$.

Modulo 7 to accommodate at least 6 packets.

Linear equations:

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 1 \pmod 7 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 7 \\ P(3) = 2a_2 + 3a_1 + a_0 &\equiv 4 \pmod 7 \end{aligned}$$

$6a_1 + 3a_0 = 2 \pmod 7$, $5a_1 + 4a_0 = 0 \pmod 7$

$a_1 = 2a_0$. $a_0 = 2 \pmod 7$ $a_1 = 4 \pmod 7$ $a_2 = 2 \pmod 7$

$P(x) = 2x^2 + 4x + 2$

$P(1) = 1$, $P(2) = 4$, and $P(3) = 4$

Send
Packets: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

Notice that packets contain "x-values".

Polynomials.

- ..give Secret Sharing.
- ..give Erasure Codes.

Error Correction:

Noisy Channel: corrupts k packets. (rather than loss.)

Additional Challenge: Finding which packets are corrupt.

Bad reception!

Send: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

Recieve: (1,1) (3,4), (6,0)
Reconstruct?

Format: $(i, R(i))$.

Lagrange or linear equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 1 \pmod 7 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 7 \\ P(6) = 2a_2 + 3a_1 + a_0 &\equiv 0 \pmod 7 \end{aligned}$$

Channeling Sahai ...

$P(x) = 2x^2 + 4x + 2$

Message? $P(1) = 1, P(2) = 4, P(3) = 4$.

Questions for Review

You want to encode a secret consisting of 1,4,4.

How big should modulus be?
Larger than 144 and prime!

You want to send a message consisting of packets 1,4,2,3,0

through a noisy channel that loses 3 packets.

How big should modulus be?
Larger than 8 and prime!

Send n packets b-bit packets, with k errors.
Modulus should be larger than $n+k$ and also larger than 2^b.