Today.

Finish Welsh-Berlekamp.

Countability.

The Scheme.

Problem: Communicate n packets $m_1,\ldots,m_n$ on noisy channel that corrupts $\leq k$ packets.

Reed-Solomon Code:

1. Make a polynomial, $P(x)$ of degree $n-1$, that encodes message.
   - $P(1)=m_1,\ldots,P(n)=m_n$.
   - Comment: could encode with packets as coefficients.
2. Send $P(1),\ldots,P(n+2k)$.

After noisy channel: Recieve values $R(1),\ldots,R(n+2k)$.

Properties:
(1) $P(i)=R(i)$ for at least $n+k$ points i,
(2) $P(x)$ is unique degree $n-1$ polynomial that contains $\geq n+k$ received points.

Slow solution.

Brute Force:
For each subset of $n+k$ points
Fit degree $n-1$ polynomial, $Q(x)$, to n of them.
Check if consistent with $n+k$ of the total points.
If yes, output $Q(x)$.

- For subset of $n+k$ pts where $R(i)=P(i)$, method will reconstruct $P(x)$!
- For any subset of $n+k$ pts,
  1. there is unique degree $n-1$ polynomial $Q(x)$ that fits n of them
  2. and where $Q(x)$ is consistent with $n+k$ points $\implies P(x)=Q(x)$.

Reconstructs $P(x)$ and only $P(x)$!!

Error Locater Polynomial.

$$\begin{aligned}
E(1)(p_{n-1}+\cdots p_0) &\equiv R(1)E(1) \pmod p\\
&\vdots\\
E(i)(p_{n-1}i^{n-1}+\cdots p_0) &\equiv R(i)E(i) \pmod p\\
&\vdots\\
E(m)(p_{n-1}(n+2k)^{n-1}+\cdots p_0) &\equiv R(m)E(m) \pmod p
\end{aligned}$$

...so satisfied, I'm on my way.

$m=n+2k$ satisfied equations, $n+k$ unknowns. But nonlinear!

We have
$Q(x)=E(x)P(x)=a_{n+k-1}x^{n+k-1}+\cdots a_0$.
and
$E(x)=x^k+b_{k-1}x^{k-1}\cdots b_0$.

Equations:

$$Q(i)=R(i)E(i).$$

and linear in a_i and coefficients of b_j!

Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree k ...
  $$E(x)=x^k+b_{k-1}x^{k-1}\cdots b_0.$$
  $\implies k$ (unknown) coefficients. Leading coefficient is 1.
- $Q(x)=P(x)E(x)$ has degree $n+k-1$ ...
  $$Q(x)=a_{n+k-1}x^{n+k-1}+a_{n+k-2}x^{n+k-2}+\cdots a_0$$
  $\implies n+k$ (unknown) coefficients.

Number of unknown coefficients: $n+2k$.

Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1,\ldots,i,n+2k=m$,

$$Q(i)=R(i)E(i) \pmod p$$

Gives $n+2k$ linear equations.

$$\begin{aligned}
a_{n+k-1}+\ldots a_0 &\equiv R(1)(1+b_{k-1}\cdots b_0) \pmod p\\
a_{n+k-1}(2)^{n+k-1}+\ldots a_0 &\equiv R(2)((2)^k+b_{k-1}(2)^{k-1}\cdots b_0) \pmod p\\
&\vdots\\
a_{n+k-1}(m)^{n+k-1}+\ldots a_0 &\equiv R(m)((m)^k+b_{k-1}(m)^{k-1}\cdots b_0) \pmod p
\end{aligned}$$

..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x)=Q(x)/E(x)$.

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

$Q(x) = E(x)P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$E(x) = x - b_0$

$Q(i) = R(i)E(i)$.

$$
\begin{aligned}
a_3 + a_2 + a_1 + a_0 &\equiv 3(1 - b_0) \pmod 7\\
a_3 + 4a_2 + 2a_1 + a_0 &\equiv 1(2 - b_0) \pmod 7\\
6a_3 + 2a_2 + 3a_1 + a_0 &\equiv 6(3 - b_0) \pmod 7\\
a_3 + 2a_2 + 4a_1 + a_0 &\equiv 0(4 - b_0) \pmod 7\\
6a_3 + 4a_2 + 5a_1 + a_0 &\equiv 3(5 - b_0) \pmod 7
\end{aligned}
$$

$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$ and $b_0 = 2$.

$Q(x) = x^3 + 6x^2 + 6x + 5$.

$E(x) = x - 2$.

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5$.
$E(x) = x - 2$.

```
                 1 x^2 + 1 x + 1
          ----------------
    x - 2 ) x^3  + 6 x^2 + 6 x + 5
            x^3  - 2 x^2
            ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
                           -----
                               0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6$.

What is $\frac{x-2}{x-2}$? 1
Except at $x = 2$? Hole there?

Error Correction: Berlekamp-Welsh

Message: $m_1, \ldots, m_n$.
Sender:

1. Form degree $n-1$ polynomial $P(x)$ where $P(i) = m_i$.
2. Send $P(1), \ldots, P(n+2k)$.

Receiver:

1. Receive $R(1), \ldots, R(n+2k)$.
2. Solve $n+2k$ equations, $Q(i) = E(i)R(i)$ to find $Q(x) = E(x)P(x)$ and $E(x)$.
3. Compute $P(x) = Q(x)/E(x)$.
4. Compute $P(1), \ldots, P(n)$.

Check your undersanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.

Efficiency? Sure. Only $n+2k$ values.
See where it is 0.

Hmmm...

Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

Existence: there is a $P(x)$ and $E(x)$ that satisfy equations.

Unique solution for $P(x)$

Uniqueness: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). \quad (1)$$

Proof:
We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \quad (2)$$

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
and agree on $n+2k$ points
$E(x)$ and $E'(x)$ have at most k zeros each.
Can cross divide at n points.
$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}$ equal on n points.
Both degree $\leq n \implies$ Same polynomial! $\square$

Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \dots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).$$

Cross multiplying gives equality in fact for these points. $\square$

Points to polynomials, have to deal with zeros!

Example: dealing with $\frac{x-2}{x-2}$ at $x = 2$.

Yaaay!!!!

Berlekamp-Welsh algorithm decodes correctly when k errors!

Quick Check. Error Correction.

Communicate n packets, with k erasures.

How many packets? $n+k$
How to encode? With polynomial, $P(x)$.
Of degree? $n-1$
Recover? Reconstruct $P(x)$ with any n points!

Communicate n packets, with k errors.

How many packets? $n+2k$
Why?
k changes to make diff. messages overlap
How to encode? With polynomial, $P(x)$. Of degree? $n-1$.
Recover?
Reconstruct error polynomial, $E(X)$, and $P(x)$!
Nonlinear equations.
Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes. Welsh-Berlekamp Decoding. Perfection!

Next up: how big is infinity.

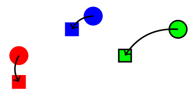
- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

Infinite!

Is one bigger or smaller?

Same size?



Same number?
Make a function f : Circles $\to$ Squares.
f(red circle) $=$ red square
f(blue circle) $=$ blue square
f(circle with black border) $=$ square with black border
One to one. Each circle mapped to different square.
One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
Onto. Each square mapped to from some circle .
Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

Isomorphism principle: If there is $f : D \to R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \to R$.
One to One:
For all $\forall x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.
or
$\forall x, y \in D$, $f(x) = f(y) \implies x = y$.

Onto: For all $y \in R$, $\exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \to R$ then $|D| = |R|$.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Natural numbers. $0, 1, 2, 3, \ldots$.

Positive integers. $1, 2, 3, \ldots$.

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.
One to one!

For any natural number n, for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.
Onto for $\mathbb{N}$

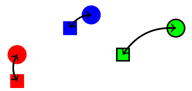
Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

But.. but Where's zero? "Comes from 1."

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.
$f(n) = n + 1$. $0 \to 1, 1 \to 2, \ldots$

Bijection from A to $B \implies$ a bijection from B to A.



Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

$f : N \to E$.

$f(n) \to 2n$.

Onto: $\forall e \in E$, $f(e/2) = e$. $e/2$ is natural since e is even
One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?
Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$
if x is even and y is even,
then $x/2 \neq y/2 \implies f(x) \neq f(y)$
....

Onto: For any $z \in Z$,
if $z \geq 0$, $f(2z) = z$ and $2z \in N$.
if $z < 0$, $f(2|z| - 1) = z$ and $2|z| + 1 \in N$.

Integers and naturals have same size!

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing "is" a bijection with a subset of natural numbers.
Function $\equiv$ "Position in list."
If finite: bijection with $\{0, \ldots, |S| - 1\}$
If infinite: bijection with N.

Enumeration example.

All binary strings.
$B = \{0,1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$.
ϕ is empty string.

For any string, it appears at some position in the list.
If n bits, it will appear before position 2^{n+1}.

Should be careful here.

$B = \{\phi;, 0, 00, 000, 0000, ...\}$
Never get to 1.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S",
"Output next element of S"
...
Any element x of S has *specific, finite* position in list.
$Z = \{0, 1, -1, 2, -2, ...\}$
$Z = \{\{0, 1, 2, \ldots,\}$ and then $\{-1, -2, \ldots\}\}$

When do you get to -1? at infinity?

Need to be careful.

61A —- streams!

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:
Get next element, x, of S,
output only if $x \in T$.

Implications:
Z^+ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.

All countably infinite sets have the same cardinality.