Counting.
Counting.
Undecidability.
Countability

Cantor Size: $A$ and $B$ have the same size if there is a bijection between $A$ and $B$. 

A set is countable if there is a bijection from $A$ to a subset of the natural numbers. 

A set is countably infinite if it is countable and not finite. 

A set is countable if one can enumerate it; output elements in a list where every element appears in the list in a specific finite position. 

$0, 1, -1, 2, -2, ...$ is countable. 

$\phi, \phi_0, \phi_00, \phi_01, \phi_10, \phi_11, ...$ Binary strings are countable. 

Be careful: $0, 00, 000, 0000, ...$ never gets to $1$! 

Computer Science: Stream interleaving. 

The union of two countable sets is countable!
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More fractions?

Are rationals countable.
More fractions?

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Enumerate the rational numbers in order...
More fractions?

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0, ..., 1/2, ..
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Where is 1/2 in list?
More fractions?

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After 1/3, which is after 1/4, which is after 1/5...
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Can’t even get to “next” fraction!
Can’t list in “order”.

Pairs of natural numbers.

Consider pairs of natural numbers: \( N \times N \)
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E.g.: $(1, 2), (100, 30)$, etc.
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Consider pairs of natural numbers: $N \times N$
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Enumerate in list:

$(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$, $(0,2)$, $\ldots$

The pair $(a, b)$ is within the first $(a+b)^2$ elements of list!

(i.e., "square containing triangle ".)
Pairs of natural numbers.

Enumerate in list:
(0, 0),
(1, 0),
(0, 1),
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The pair (a, b), is within the first \((a + b)(a + b)\) elements of list (i.e., “square containing triangle ”).

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
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Positive rational number.
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Positive rational number.
Lowest terms: $a/b$
Rationals?

Positive rational number.
Lowest terms: \( \frac{a}{b} \)
\( a, b \in N \)
Rationals?

Positive rational number. Lowest terms: $a/b$

$a, b \in N$

with $gcd(a, b) = 1$. 

Infinite subset of $\mathbb{N} \times \mathbb{N}$. Countably infinite!

All rational numbers? Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list. First negative, then nonegative. No!

Repeatedly and alternatively take one from each list. Interleave Streams in 61A

The rationals are countably infinite.
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Interleave Streams in 61A

The rationals are countably infinite.
Real numbers are same size as integers?
The reals.

Are the set of reals countable?
The reals.

Are the set of reals countable?
Lets consider the reals [0, 1].
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Each real has a decimal representation.
The reals.

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$.785398162...$
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$.500000000...$ (1/2)

$.785398162...$ $\pi/4$
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- $.345212312...$ Some real number
Diagonalization.

If countable, there a listing, \( L \) contains all reals.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example 0: .500000000...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example:

0: .500000000...
1: .785398162...

... Construct “diagonal” number:

... 7 7 6 7 7 ...

Diagonal Number:

Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

6 is apparently isn’t afraid!

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0, 1]$ is not countable!!
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example
0: .500000000...
1: .785398162...
2: .367879441...

Construct “diagonal” number:
.7767...

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Diagonalization.

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0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...

Construct "diagonal" number:

.7
7
6
7
7...

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3: .632120558...
4: .345212312...
Diagonalization.

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0: .500000000...
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Construct “diagonal” number:
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \textbf{.5}000000000...
1: \textbf{.7}85398162...
2: \textbf{.3}67879441...
3: \textbf{.6}32120558...
4: \textbf{.3}45212312...

Construct “diagonal” number: .7
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

Construct “diagonal” number: \( .776 \)
If countable, there a listing, \( L \) contains all reals. For example

0: \(.5000000000\ldots\)
1: \(.785398162\ldots\)
2: \(.367879441\ldots\)
3: \(.632120558\ldots\)
4: \(.345212312\ldots\)

Construct “diagonal” number: \(.7767\)
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \(.500000000...\)
1: \(.785398162...\)
2: \(.367879441...\)
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\[
\begin{array}{cccc}
0 & 5 & 0 & 0 \\
1 & 7 & 8 & 5 \\
2 & 3 & 6 & 7 \\
3 & 6 & 3 & 2 \\
4 & 3 & 4 & 5 \\
\end{array}
\]

Construct “diagonal” number: \(.77677\)
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

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... 

Construct “diagonal” number: \( .77677 \ldots \)

Diagonal Number: Digit \( i \) is 7 if number \( i \)'s \( i \)th digit is not 7
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

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4: \( .345212312\ldots \)

: 

Construct “diagonal” number: \( .77677\ldots \)

Diagonal Number: Digit \( i \) is 7 if number \( i \)'s \( i \)th digit is not 7 and 6 otherwise.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

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3: \( .632120558 \ldots \)
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\[ \vdots \]

Construct “diagonal” number: \( .77677 \ldots \)

**Diagonal Number:** Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7 and 6 otherwise. **6 is apparently isn’t afraid!**
Diagonalization.

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Diagonal number for a list differs from every number in list!
If countable, there a listing, \( L \) contains all reals. For example

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Construct “diagonal” number: \(.77677...\)

Diagonal Number: Digit \( i \) is 7 if number \( i \)'s \( i \)th digit is not 7 and 6 otherwise. \( 6 \) is apparently isn’t afraid!

Diagonal number for a list differs from every number in list! Diagonal number not in list.
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

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1: \( .785398162\ldots \)
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Diagonal number not in list.

Diagonal number is real.
Diagonalization.

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Diagonal number is real.
Contradiction!
Diagonalization.

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Diagonal number is real.
Contradiction!

Subset \([0, 1]\) is not countable!!
All reals?

Subset [0, 1] is not countable!!
All reals?

Subset $[0, 1]$ is not countable!!
What about all reals?
All reals?

Subset [0, 1] is not countable!!

What about all reals?
No.
All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
If reals are countable then so is $[0, 1]$. 
1. Assume that a set $S$ can be enumerated.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$. 
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$.
4. Show that $t$ is different from all elements in the list $\Rightarrow t$ is not in the list.
5. Show that $t$ is in $S$.
6. Contradiction.
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Another diagonalization.

The set of all subsets of \( N \).
Another diagonalization.

The set of all subsets of $\mathbb{N}$.
Example subsets of $\mathbb{N}$:
Another diagonalization.

The set of all subsets of $\mathbb{N}$.
Example subsets of $\mathbb{N}$: \{0\},
Another diagonalization.

The set of all subsets of $\mathbb{N}$.
Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$,
Another diagonalization.

The set of all subsets of \( N \).
Example subsets of \( N \): \( \{0\} \), \( \{0, \ldots, 7\} \), evens,
Another diagonalization.

The set of all subsets of $N$.
Example subsets of $N$: \{0\}, \{0, \ldots, 7\}, evens, odds,
Another diagonalization.

The set of all subsets of \( N \).
Example subsets of \( N \): \( \{0\} \), \( \{0,\ldots,7\} \), evens, odds, primes,
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The set of all subsets of $\mathbb{N}$.
Example subsets of $\mathbb{N}$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds, primes, ...
Another diagonalization.

The set of all subsets of \( N \).
Example subsets of \( N \): \( \{0\} \), \( \{0,\ldots,7\} \), evens, odds, primes, ...
Assume that it is countable.
Another diagonalization.

The set of all subsets of $N$.
Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes, ...
Assume that it is countable.
There is a listing, $L$, that contains all subsets of $N$. 
Another diagonalization.

The set of all subsets of $N$.
  Example subsets of $N$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds, primes, ...

Assume that it is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
Another diagonalization.

The set of all subsets of $N$.  
Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes, ...

Assume that it is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.  

Contradiction.

Theorem: The set of all subsets of $N$ is not countable.  
(The set of all subsets of $S$, is the powerset of $N$.)

Another diagonalization.

The set of all subsets of $\mathbb{N}$.
Example subsets of $\mathbb{N}$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds, primes, ...

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Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes, ...

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If $i$th set in $L$ does not contain $i$, $i \in D$.
   otherwise $i \notin D$.

$D$ is different from $i$th set in $L$ for every $i$. 


Another diagonalization.

The set of all subsets of $N$.
   Example subsets of $N$: \{0\}, \{0, \ldots, 7\}, evens, odds, primes, ...

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\[ \implies D \text{ is not in the listing.} \]
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   $\implies D$ is not in the listing.

$D$ is a subset of $N$. 
Another diagonalization.

The set of all subsets of \( N \).
Example subsets of \( N \): \( \{0\} \), \( \{0,\ldots,7\} \), evens, odds, primes, ...

Assume that it is countable.

There is a listing, \( L \), that contains all subsets of \( N \).

Define a diagonal set, \( D \):
If \( i \)th set in \( L \) does not contain \( i \), \( i \in D \).
otherwise \( i \notin D \).

\( D \) is different from \( i \)th set in \( L \) for every \( i \).
\[ \implies \] \( D \) is not in the listing.

\( D \) is a subset of \( N \). \( L \) does not contain all subsets of \( N \).
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0,\ldots,7\} \), evens, odds, primes, ...  

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Contradiction.
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**Theorem:** The set of all subsets of \( N \) is not countable.
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes, ...

Assume that it is countable.

There is a listing, $L$, that contains all subsets of $N$.

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$D$ is different from $i$th set in $L$ for every $i$.

$\implies D$ is not in the listing.

$D$ is a subset of $N$. $L$ does not contain all subsets of $N$.

Contradiction.

Theorem: The set of all subsets of $N$ is not countable.
(The set of all subsets of $S$, is the powerset of $N$.)
Diagonalize Natural Number.

Natural numbers have a listing, \( L \).
Diagonalize Natural Number.

Natural numbers have a listing, $L$.

Make a diagonal number, $D$:

- differ from $i$th element of $L$ in $i$th digit.
Diagonalize Natural Number.

Natural numbers have a listing, $L$.

Make a diagonal number, $D$:
   differ from $i$th element of $L$ in $i$th digit.

Differs from all elements of listing.
Diagonalize Natural Number.

Natural numbers have a listing, \( L \).

Make a diagonal number, \( D \):
   differ from \( i \)th element of \( L \) in \( i \)th digit.

Differs from all elements of listing.

\( D \) is a natural number...
Diagonalize Natural Number.

Natural numbers have a listing, $L$.

Make a diagonal number, $D$:

- differ from $i$th element of $L$ in $i$th digit.

Differs from all elements of listing.

$D$ is a natural number... Not.
Natural numbers have a listing, $L$.
Make a diagonal number, $D$:
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$D$ is a natural number... Not.
Any natural number has a finite number of digits.
Natural numbers have a listing, $L$.

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$D$ is a natural number... Not.

Any natural number has a finite number of digits.

“Construction” requires an infinite number of digits.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

First of Hilbert’s problems!
Cardinalities of uncountable sets?

Cardinality of \((0, 1]\) smaller than all the reals?
Cardinalities of uncountable sets?

Cardinality of $(0, 1]$ smaller than all the reals?

$f : R^+ \rightarrow (0, 1]$. 
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : \mathbb{R}^+ \rightarrow (0, 1]$. 

$$f(x) = \begin{cases} 
  x + \frac{1}{2} & 0 \leq x \leq 1/2 \\
  \frac{1}{4x} & x > 1/2 
\end{cases}$$
Cardinalities of uncountable sets?

Cardinality of \((0, 1]\) smaller than all the reals?

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\]

One to one.

Onto. Every element in \((0, 1]\) has pre-image.
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One to one. \(x \neq y\)
Cardinalities of uncountable sets?

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If both in \([0, 1/2]\),
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One to one. $x \neq y$

If both in $[0,1/2]$, a shift
Cardinalities of uncountable sets?

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If both in \([0, 1/2]\), a shift \(\implies f(x) \neq f(y)\).
Cardinalities of uncountable sets?

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One to one. \(x \neq y\)
If both in \([0, 1/2]\), a shift \(\implies f(x) \neq f(y)\).
If neither in \([0, 1/2]\)
Cardinalities of uncountable sets?

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One to one. \(x \neq y\)
If both in \([0, 1/2]\), a shift \(\implies f(x) \neq f(y)\).
If neither in \([0, 1/2]\) a division
Cardinalities of uncountable sets?

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If one is in \([0, 1/2]\) and one isn’t,
Cardinalities of uncountable sets?

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If one is in \([0, 1/2]\) and one isn’t, different ranges
Cardinalities of uncountable sets?

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One to one. \(x \neq y\)
If both in \([0, 1/2]\), a shift \(\Rightarrow f(x) \neq f(y)\).
If neither in \([0, 1/2]\) a division \(\Rightarrow f(x) \neq f(y)\).
If one is in \([0, 1/2]\) and one isn’t, different ranges \(\Rightarrow f(x) \neq f(y)\).
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If one is in \([0,1/2]\) and one isn’t, different ranges \( \implies f(x) \neq f(y). \)

Onto. Every element in \((0,1]\) has pre-image.
Cardinalities of uncountable sets?

Cardinality of \((0, 1]\) smaller than all the reals?

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If both in \([0, 1/2]\), a shift \( \implies f(x) \neq f(y) \).
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Bijection!
Cardinalities of uncountable sets?

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One to one. $x \neq y$
If both in $[0, 1/2]$, a shift $\implies f(x) \neq f(y)$.
If neither in $[0, 1/2]$ a division $\implies f(x) \neq f(y)$.
If one is in $[0, 1/2]$ and one isn’t, different ranges $\implies f(x) \neq f(y)$.

Onto. Every element in $(0, 1]$ has pre-image.

Bijection!
Cardinalities of uncountable sets?

Cardinality of \((0, 1]\) smaller than all the reals?

\[
f : R^+ \to (0, 1].
\]

\[
f(x) = \begin{cases} 
  x + \frac{1}{2} & 0 \leq x \leq 1/2 \\
  \frac{1}{4x} & x > 1/2 
\end{cases}
\]

One to one. \(x \neq y\)

If both in \([0, 1/2]\), a shift \(\implies f(x) \neq f(y)\).

If neither in \([0, 1/2]\) a division \(\implies f(x) \neq f(y)\).

If one is in \([0, 1/2]\) and one isn’t, different ranges \(\implies f(x) \neq f(y)\).

Onto. Every element in \((0, 1]\) has pre-image.

Bijection!

\([0, 1]\) is same cardinality as nonegative reals.
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.
There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.
Resolution of hypothesis?

Gödel. 1940.

Can't use math!
If math doesn't contain a contradiction.
This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Uh oh....
Resolution of hypothesis?

Gödel. 1940.
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Uh oh....
Next Topic: Undecidability.

- Undecidability.
Barber paradox.

Barber announces:
Barber paradox.

Barber announces:
“The barber shaves every person who does not shave themselves.”
Barber paradox.

Barber announces:
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Who shaves the barber?
Barber paradox.

Barber announces:
“The barber shaves every person who does not shave themselves.”

Who shaves the barber?

Get around paradox?
Barber paradox.

Barber announces:
“The barber shaves every person who does not shave themselves.”

Who shaves the barber?

Get around paradox?
The barber lies.
Russell’s Paradox.

Naive Set Theory: Any definable collection is a set.
Russell’s Paradox.

Naive Set Theory: Any definable collection is a set.

$$\exists y \forall x (x \in y \iff P(x)) \quad (1)$$
Russell’s Paradox.

Naive Set Theory: Any definable collection is a set.

\[ \exists y \forall x (x \in y \iff P(x)) \quad (1) \]

\( y \) is the set of elements that satisfies the proposition \( P(x) \).
Russell’s Paradox.

Naive Set Theory: Any definable collection is a set.

$$\exists y \forall x (x \in y \iff P(x))$$  \hspace{1cm} (1)

$y$ is the set of elements that satisfies the proposition $P(x)$.

$P(x) = x \notin x$.
Russell’s Paradox.

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\[ \exists y \forall x (x \in y \iff P(x)) \]  \hspace{1cm} (1)

\( y \) is the set of elements that satisfies the proposition \( P(x) \).

\( P(x) = x \notin x \).

There exists a \( y \) that satisfies statement 1 for \( P(\cdot) \).
Russell’s Paradox.

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$$\exists y \forall x (x \in y \iff P(x)) \quad (1)$$

$y$ is the set of elements that satisfies the proposition $P(x)$.

$P(x) = x \notin x$.

There exists a $y$ that satisfies statement 1 for $P(\cdot)$.

Take $x = y$. 
Russell’s Paradox.

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Oops!
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Oops!

What type of object is a set that contain sets?
Russell’s Paradox.

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There exists a \( y \) that satisfies statement 1 for \( P(\cdot) \).

Take \( x = y \).

\[ y \in y \iff y \not\in y. \]

Oops!

What type of object is a set that contain sets?

Axioms changed.
Changing Axioms?

Goedel: Any set of axioms is either

Concrete example:
Continuum hypothesis: "no cardinality between reals and naturals."
Continuum hypothesis not disprovable in ZFC (Goedel 1940.)
Continuum hypothesis not provable. (Cohen 1963: only Fields medal in logic)

BTW: Cantor... bipolar disorder...
Goedel... starved himself out of fear of being poisoned...
Russell... was fine... but for... two schizophrenic children...
Dangerous work? See Logicomix by Doxiaidis, Papadimitriou (professor here?), Papadatos, Di Donna.
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Is it actually useful?

Write me a program checker!
Is it actually useful?

Write me a program checker!
Check that the compiler works!
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How about.. Check that the compiler terminates on a certain input.
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$HALT(P, I)$
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- $P$ - program
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Determines if $P(I)$ ($P$ run on $I$) halts or loops forever.

Notice: Need a computer...with the notion of a stored program!!!! (not an adding machine! not a person and an adding machine.)
Program is a text string. Text string can be an input to a program. Program can be an input to a program.
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Implementing HALT.
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\[ \text{HALT}(P, I) \]
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$HALT(P, I)$

$P$ - program
Implementing HALT.

\[ \text{HALT}(P, I) \]

- \( P \) - program
- \( I \) - input.
Implementing HALT.

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\[ I \text{ - input.} \]

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Implementing HALT.

\[
HALT(P, I)
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- \(P\) - program
- \(I\) - input.

Determines if \(P(I)\) (\(P\) run on \(I\)) halts or loops forever.

Run \(P\) on \(I\) and check!
Implementing HALT.

$HALT(P, I)$
- $P$ - program
- $I$ - input.

Determines if $P(I)$ ($P$ run on $I$) halts or loops forever.

Run $P$ on $I$ and check!

How long do you wait?
Implementing HALT.

\[ \text{HALT}(P, I) \]
\[ P - \text{program} \]
\[ I - \text{input.} \]

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.
Run \( P \) on \( I \) and check!
How long do you wait?
Something about infinity here, maybe?
Halt does not exist.
Halt does not exist.

\[ \text{HALT}(P, I) \]
Halt does not exist.

\[ \text{HALT}(P, I) \]

\[ P \text{- program} \]
Halt does not exist.

\[ \text{HALT}(P, I) \]

- \(P\) - program
- \(I\) - input.
Halt does not exist.

HALT$(P, I)$

$P$ - program

$I$ - input.

Determines if $P(I)$ ($P$ run on $I$) halts or loops forever.
Halt does not exist.

\[ \text{HALT}(P, I) \]
- \( P \) - program
- \( I \) - input.

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.
Halt does not exist.

\[ \text{HALT}(P, I) \]

- \( P \) - program
- \( I \) - input.

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes!
Halt does not exist.

\[ \text{HALT}(P, I) \]

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- \( I \) - input.

 Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes! No!
Halt does not exist.

\[ HALT(P, I) \]

\[ P \] - program

\[ I \] - input.

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program \( HALT \).

**Proof:** Yes! No! Yes!

What is Rao talking about?

(A) Rao is confused.

(B) Fermat's Theorem.

(C) Diagonalization.

(C). maybe (A) too.
Halt does not exist.

\[ \text{HALT}(P, I) \]

- \( P \) - program
- \( I \) - input.

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program \( \text{HALT} \).

**Proof:** Yes! No! Yes! No!
Halt does not exist.

\[ \text{HALT}(P, I) \]

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Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program \( \text{HALT} \).

**Proof:** Yes! No! Yes! No! No!
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\[ \text{HALT}(P, I) \]
\[ P \text{ - program} \]
\[ I \text{ - input.} \]

Determines if \( P(I) \) (\( P \) run on \( I \)) halts or loops forever.

**Theorem:** There is no program HALT.

**Proof:** Yes! No! Yes! No! No! Yes!
Halt does not exist.

$HALT(P, I)$

- $P$ - program
- $I$ - input.

Determines if $P(I)$ ($P$ run on $I$) halts or loops forever.

**Theorem:** There is no program $HALT$.

**Proof:** Yes! No! Yes! No! No! Yes! No!
Halt does not exist.

\[ \text{HALT}(P, I) \]
- \( P \) - program
- \( I \) - input.

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**Theorem:** There is no program HALT.

**Proof:** Yes! No! Yes! No! No! Yes! No! Yes!
Halt does not exist.

\[ HALT(P, I) \]
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What is Rao talking about?

(A) Rao is confused.
(B) Fermat’s Theorem.
(C) Diagonalization.

(C). maybe (A) too.
Halt and Turing.

Proof:

Assume there is a program \( \text{HALT} \) \((\cdot, \cdot)\).

1. If \( \text{Turing}(P) \) halts, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program \( \text{HALT} \).

There is text that "is" the program \( \text{Turing} \).

See above!

Can run Turing on Turing!

Does \( \text{Turing}(\text{Turing}) \) halt?

Case 1: \( \text{Turing}(\text{Turing}) \) halts

\( \Rightarrow \)

\( \text{HALTS}(\text{Turing}, \text{Turing}) = \text{halts} \)

\( \Rightarrow \)

\( \text{Turing}(\text{Turing}) \) loops forever.

Case 2: \( \text{Turing}(\text{Turing}) \) loops forever

\( \Rightarrow \)

\( \text{HALTS}(\text{Turing}, \text{Turing}) \neq \text{halts} \)

\( \Rightarrow \)

\( \text{Turing}(\text{Turing}) \) halts.

Contradiction.

Program \( \text{HALT} \) does not exist!

Questions?
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$. 

Case 1: $Turing(Turing)$ halts $= \Rightarrow$ then $HALT(Turing, Turing) = halts = \Rightarrow$ $Turing(Turing)$ loops forever.

Case 2: $Turing(Turing)$ loops forever $= \Rightarrow$ then $HALT(Turing, Turing) \neq halts = \Rightarrow$ $Turing(Turing)$ halts. 

Contradiction.

Program $HALT$ does not exist!
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

Turing($P$)
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

$Turing(P)$
1. If $HALT(P,P) =$“halts”, then go into an infinite loop.
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

Turing($P$)
1. If $HALT(P, P) = \text{halts}$, then go into an infinite loop.
2. Otherwise, halt immediately.
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

**Turing(P)**
1. If $HALT(P,P) = \text{"halts"}$, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT.
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

Turing($P$)
1. If $HALT(P, P) =$“halts”, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $HALT$.
There is text that “is” the program $HALT$. 
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

$\text{Turing}(P)$

1. If $HALT(P,P) = \text{"halts"}$, then go into an infinite loop.
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There is text that "is" the program $HALT$.
There is text that is the program $Turing$. 

---
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1. If $HALT(P,P) = \text{"halts"}$, then go into an infinite loop.
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There is text that “is” the program HALT.
There is text that is the program Turing. **See above!**
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Assumption: there is a program HALT. 
There is text that “is” the program HALT. 
There is text that is the program Turing. See above!

Can run Turing on Turing!
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

$\text{Turing}(P)$
1. If $HALT(P,P) =$“halts”, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $HALT$. There is text that “is” the program $HALT$. There is text that is the program $Turing$. See above! Can run $Turing$ on $Turing$!

Does $Turing(Turing)$ halt?
Halt and Turing.

**Proof:** Assume there is a program \( \text{HALT}(\cdot, \cdot) \).

**Turing(P)**

1. If \( \text{HALT}(P, P) = \text{"halts"} \), then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT.

There is text that “is” the program HALT.
There is text that is the program Turing. **See above!**
Can run Turing on Turing!

Does **Turing(Turing)** halt?

Case 1: Turing(Turing) halts
Halt and Turing.

Proof: Assume there is a program \( HALT(\cdot, \cdot) \).

\begin{itemize}
  \item Turing(P)
    \begin{enumerate}
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      \item Otherwise, halt immediately.
    \end{enumerate}
\end{itemize}

Assumption: there is a program HALT.
There is text that “is” the program HALT.
There is text that is the program Turing. See above!
Can run Turing on Turing!

Does Turing(Turing) halt?

Case 1: Turing(Turing) halts
  \( \implies \) then \( HALTS(Turing, Turing) = \text{halts} \)
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

$Turing(P)$
1. If $HALT(P,P) =$“halts”, then go into an infinite loop.
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Can run $Turing$ on $Turing$!

Does $Turing(Turing)$ halt?

Case 1: $Turing(Turing)$ halts
$\implies$ then $HALTS(Turing, Turing) =$ halts
$\implies$ $Turing(Turing)$ loops forever.
Halt and Turing.

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1. If $HALT(P, P) = \text{"halts"}$, then go into an infinite loop.
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Case 1: $Turing(Turing)$ halts
⇒ then $HALTS(Turing, Turing) = \text{halts}$
⇒ $Turing(Turing)$ loops forever.

Case 2: $Turing(Turing)$ loops forever
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

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Case 1: $Turing(Turing)$ halts
\[ \implies \text{then } HALTS(Turing, Turing) = \text{halts} \]
\[ \implies Turing(Turing) \text{ loops forever.} \]

Case 2: $Turing(Turing)$ loops forever
\[ \implies \text{then } HALTS(Turing, Turing) \neq \text{halts} \]
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

Turing($P$)
1. If $HALT(P,P) = \text{"halts"}$, then go into an infinite loop.
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Assumption: there is a program $HALT$.
There is text that “is” the program $HALT$.
There is text that is the program Turing. See above!
Can run Turing on Turing!

Does Turing(Turing) halt?

**Case 1:** Turing(Turing) halts
\[ \implies \text{then } HALTS(Turing, Turing) = \text{halts} \]
\[ \implies \text{Turing(Turing) loops forever.} \]

**Case 2:** Turing(Turing) loops forever
\[ \implies \text{then } HALTS(Turing, Turing) \neq \text{halts} \]
\[ \implies \text{Turing(Turing) halts.} \]
Halt and Turing.

**Proof:** Assume there is a program \( HALT(\cdot, \cdot) \).

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1. If \( HALT(P, P) = \text{"halts"} \), then go into an infinite loop.  
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Assumption: there is a program HALT.  
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Can run Turing on Turing!

Does \( Turing(Turing) \) halt?

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\[ \implies Turing(Turing) \text{ loops forever.} \]

**Case 2:** \( Turing(Turing) \) loops forever  
\[ \implies \text{then } HALTS(Turing, Turing) \neq \text{halts} \]  
\[ \implies Turing(Turing) \text{ halts.} \]

Contradiction.
Halt and Turing.

Proof: Assume there is a program $HALT(\cdot,\cdot)$.

Turing(P)
1. If $HALT(P,P) =$“halts”, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT. There is text that “is” the program HALT. There is text that is the program Turing. See above! Can run Turing on Turing!

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   $\Rightarrow$ then $HALTS(Turing, Turing) \neq \text{halts}$
   $\Rightarrow$ Turing(Turing) halts.

Contradiction. Program HALT does not exist!
**Halt and Turing.**

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$.

$\text{Turing}(P)$
1. If $HALT(P,P) \text{=} \text{“halts”}$, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $HALT$.
There is text that “is” the program $HALT$.
There is text that is the program $\text{Turing}$. *See above!* 
Can run $\text{Turing}$ on $\text{Turing}$!

Does $\text{Turing}(\text{Turing})$ halt?

Case 1: $\text{Turing}(\text{Turing})$ halts
    $\implies$ then $\text{HALTS}(\text{Turing}, \text{Turing}) \text{=} \text{halts}$
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Wow.

A lot of mind bending stuff.

See you on Friday to regroup!
Wow.

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