CS70

Counting.

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Undecidability.

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Computer Science: Stream interleaving.
The union of two countable sets is countable!

More fractions?

Are rationals countable.

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Enumerate the rational numbers in order...

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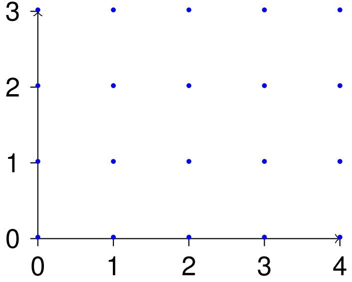

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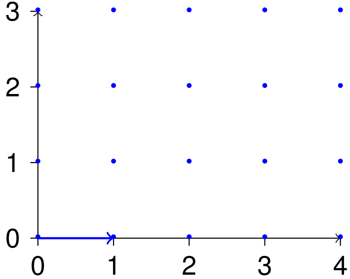

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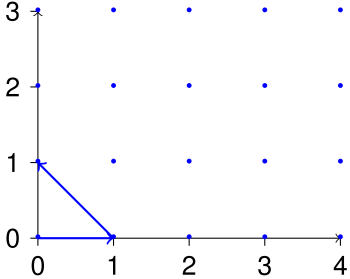

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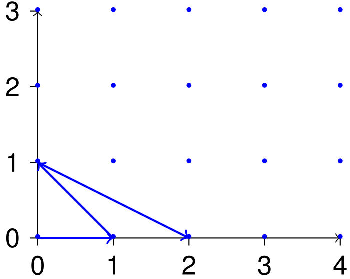

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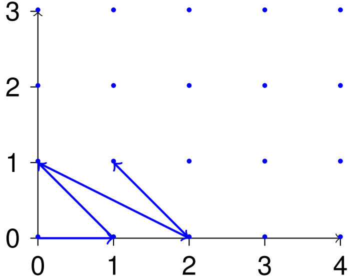

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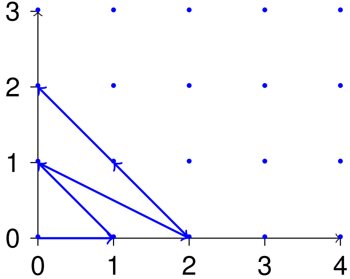

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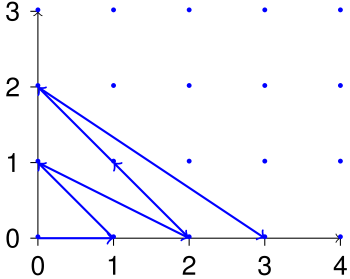

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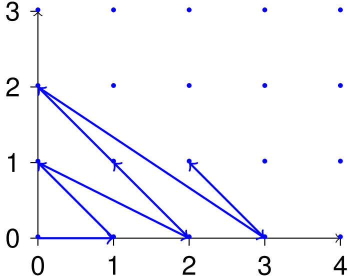

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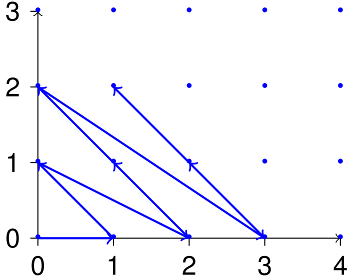

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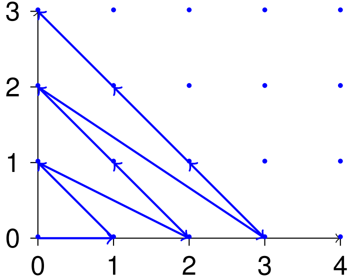

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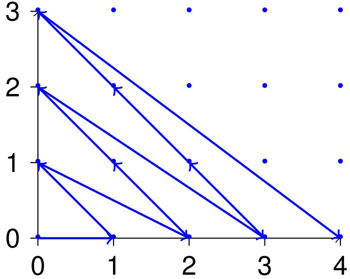

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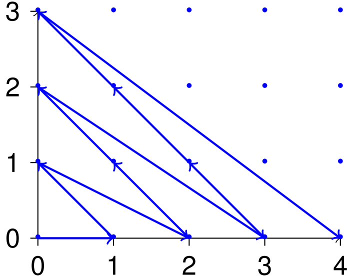

The pair (a,b), is within the first $(a+b)(a+b)$ elements of list!

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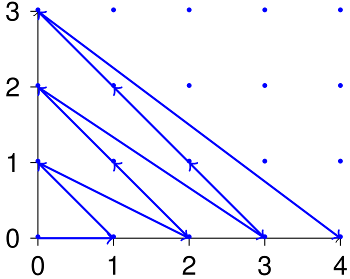

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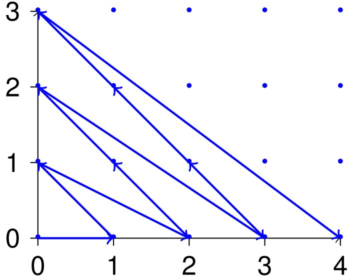

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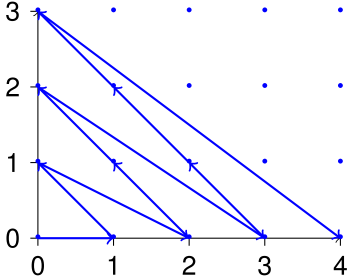

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Same size as the natural numbers!!

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Real numbers..

Real numbers are same size as integers?

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If countable, there a listing, L contains all reals.

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1: .785398162...
2: .367879441...
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⋮

Construct "diagonal" number: .77

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Contradiction!

Subset $[0,1]$ is not countable!!

All reals?

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What about all reals?

All reals?

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What about all reals?
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Any subset of a countable set is countable.

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If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

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Another diagonalization.

The set of all subsets of N.

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Example subsets of N:

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Example subsets of N: $\{0\}$,

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Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,

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There is a listing, *L*, that contains all subsets of *N*.

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Theorem: The set of all subsets of *N* is not countable.

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Theorem: The set of all subsets of *N* is not countable.
(The set of all subsets of *S*, is the **powerset** of *N*.)

Diagonalize Natural Number.

Natural numbers have a listing, L.

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Make a diagonal number, D:
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"Construction" requires an infinite number of digits.

The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.

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First of Hilbert's problems!

Cardinalities of uncountable sets?

Cardinality of $(0,1]$ smaller than all the reals?

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$$f(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

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Onto. Every element in $(0,1]$ has pre-image.

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$[0,1]$ is same cardinality as nonegative reals.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

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Can’t use math!

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Uh oh....

Next Topic: Undecidability.

- Undecidability.

Barber paradox.

Barber announces:

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Barber paradox.

Barber announces:

"The barber shaves every person who does not shave themselves."

Who shaves the barber?

Get around paradox?

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Barber announces:
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Who shaves the barber?

Get around paradox?
The barber lies.

Russell's Paradox.

Naive Set Theory: Any definable collection is a set.

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What type of object is a set that contain sets?

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Axioms changed.

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See Logicomix by Doxiaidis, Papadimitriou (professor here?), Papadatos, Di Donna.

Is it actually useful?

Write me a program checker!

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Run P on I and check!

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Something about infinity here, maybe?

Halt does not exist.

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What is Rao talking about?

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(A) Rao is confused.

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(C). maybe (A) too.

Halt and Turing.

Proof:

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Proof: Assume there is a program $HALT(\cdot, \cdot)$.

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1. If HALT(P,P) ="halts", then go into an infinite loop.
2. Otherwise, halt immediately.

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Halt and Turing.

Proof: Assume there is a program $HALT(\cdot,\cdot)$.

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1. If HALT(P,P) ="halts", then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT.
There is text that "is" the program HALT.
There is text that is the program Turing. See above!
Can run Turing on Turing!

Does Turing(Turing) halt?

Case 1: Turing(Turing) halts
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Questions?

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See you on Friday to regroup!