CS70: On to probability.

Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space

Key Points

- Uncertainty does not mean “nothing is known”
- Most real-world problems involve uncertainty
  - Predictions:
    - Will you get an A in CS70? Will the Raiders win the Super Bowl?
  - Strategy/Decision-making under uncertainty
    - Drop CS 70? How much to bet on blackjack? Buy a specific stock?
  - Engineering
    - Build a spam filter Improve wifi coverage. Control systems (Internet, airplane, robots, self-driving cars)
- How to best use ‘artificial’ uncertainty?
  - Play games of chance
  - Design randomized algorithms.
- Probability
  - Models knowledge about uncertainty: Mathematical discipline that allows you to **reason about uncertainty**.
  - Discovers best way to use that knowledge in making decisions

The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability: A precise, unambiguous, simple(!) way to think about uncertainty.

Uncertainty = Fear

Probability = Serenity

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice, practice, practice.

Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)

H

T

- Possible outcomes: Heads (H) and Tails (T)
  (*One flip yields either ‘heads’ or ‘tails’.*)
- Likelihoods: H : 50% and T : 50%

Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:

- Single coin flip: 50% chance of ‘tails’ [subjectivist]

  Willingness to bet on the outcome of a single flip

- Many coin flips: About half yield ‘tails’ [frequentist]

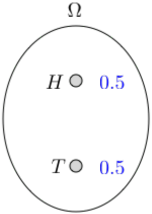
  Makes sense for many flips

- Question: Why does the fraction of tails converge to the same value every time? Statistical Regularity! Deep!

Random Experiment: Flip one Fair Coin

Flip a fair coin: model

Physical Experiment



Probability Model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set Ω of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.

Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

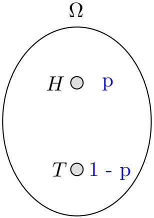
- Possible outcomes: Heads (H) and Tails (T)
- Likelihoods: $H : p \in (0, 1)$ and $T : 1 - p$
- Frequentist Interpretation:

  Flip many times $\Rightarrow$ Fraction $1 - p$ of tails
- Question: How can one figure out p? Flip many times
- Tautology? No: Statistical regularity!

Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

Physical Experiment



Probability Model

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\} \equiv \{H, T\}^2$.
- Note: $A \times B := \{(a, b) \mid a \in A, b \in B\}$ and $A^2 := A \times A$.
- Likelihoods: $1/4$ each.

Flip Glued Coins

Flips two coins glued together side by side:

Glued coins

50%

50%

- Possible outcomes: $\{HH, TT\}$.
- Likelihoods: $HH : 0.5, TT : 0.5$.
- Note: Coins are glued so that they show the same face.

Flip Glued Coins

Flips two coins glued together side by side:

Glued coins


50%


50%

- Possible outcomes: $\{HT, TH\}$.
- Likelihoods: $HT : 0.5$, $TH : 0.5$.
- Note: Coins are glued so that they show different faces.

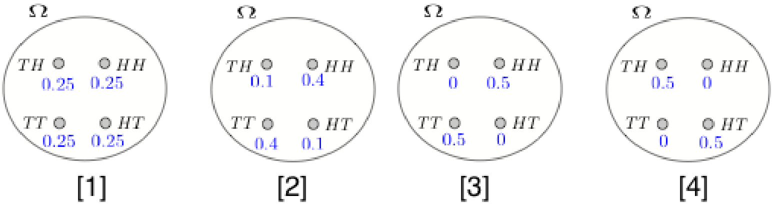
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: $\{HH, HT, TH, TT\}$.
- Likelihoods: $HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4$.
- Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.

Flipping Two Coins

Here is a way to summarize the four random experiments:

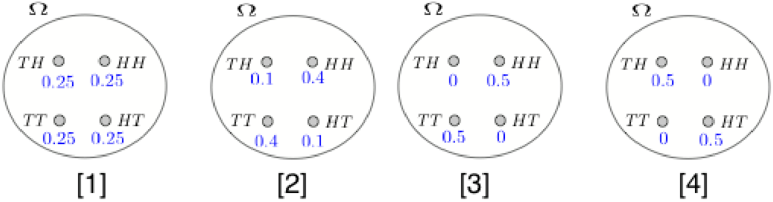


- Ω is the set of *possible* outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are ≥ 0 and add up to 1;
- Fair coins: [1]; Glued coins: $[3], [4]$;

  Spring-attached coins: [2];

Flipping Two Coins

Here is a way to summarize the four random experiments:



Important remarks:

- Each outcome describes the two coins.
- E.g., HT is one outcome of the experiment.
- It is wrong to think that the outcomes are $\{H, T\}$ and that one picks twice from that set.
- Indeed, this viewpoint misses the relationship between the two flips.
- Each $\omega \in \Omega$ describes one outcome of the complete experiment.
- Ω and the probabilities specify the random experiment.

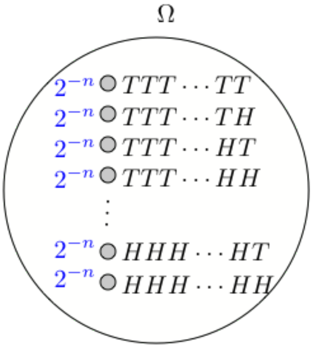
Flipping n times

Flip a fair coin n times (some $n \geq 1$):

- Possible outcomes: $\{TT\cdots T, TT\cdots H, \ldots, HH\cdots H\}$.

  Thus, 2^n possible outcomes.
- Note: $\{TT\cdots T, TT\cdots H, \ldots, HH\cdots H\} = \{H, T\}^n$.

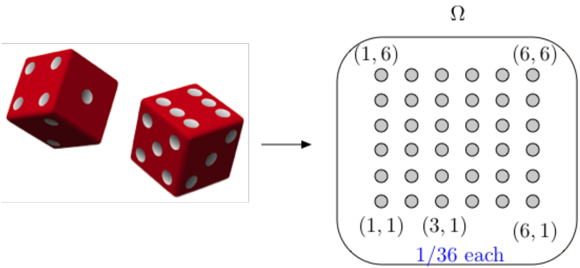
  $A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
- Likelihoods: $1/2^n$ each.



Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes: $\{1,2,3,4,5,6\}^2 = \{(a,b) \mid 1 \leq a,b \leq 6\}$.
- Likelihoods: $1/36$ for each.



Physical Experiment   Probability Model

Probability Space.

1. A "random experiment":
   (a) Flip a biased coin;
   (b) Flip two fair coins;
   (c) Deal a poker hand.

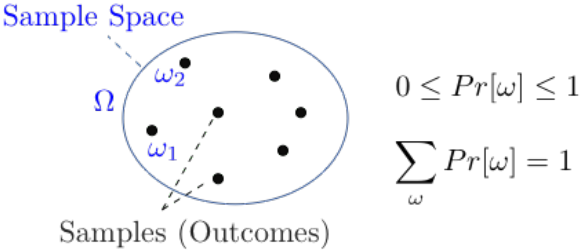
2. A set of possible outcomes: Ω.
   (a) $\Omega = \{H, T\}$;
   (b) $\Omega = \{HH, HT, TH, TT\}$; $|\Omega| = 4$;
   (c) $\Omega = \{ \underline{A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ K\spadesuit}, \underline{A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ Q\spadesuit}, \ldots\}$
   $|\Omega| = \binom{52}{5}$.

3. Assign a probability to each outcome: $Pr : \Omega \to [0,1]$.
   (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0,1]$
   (b) $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$
   (c) $Pr[\ \underline{A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ K\spadesuit}\ ] = \cdots = 1/\binom{52}{5}$

Probability Space: formalism.

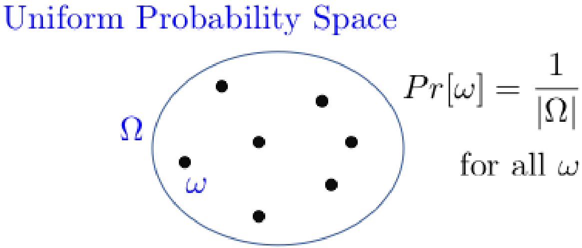
Ω is the **sample space.**
$\omega \in \Omega$ is a **sample point**. (Also called an **outcome**.)
Sample point ω has a probability $Pr[\omega]$ where

- $0 \leq Pr[\omega] \leq 1$;
- $\sum_{\omega \in \Omega} Pr[\omega] = 1$.



Probability Space: Formalism.

In a **uniform probability space** each outcome ω is equally probable: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.



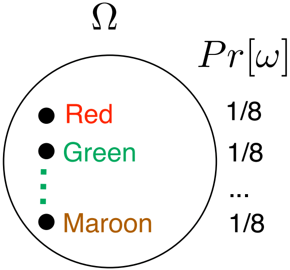
Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.

Probability Space: Formalism

Simplest physical model of a uniform probability space:

Physical experiment



Probability model

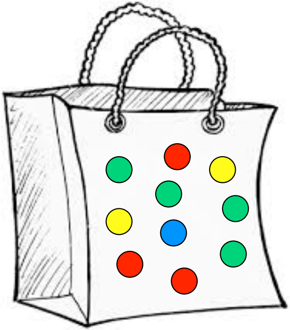
A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$

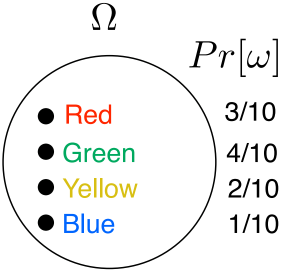
$$Pr[\text{blue}] = \frac{1}{8}.$$

Probability Space: Formalism

Simplest physical model of a non-uniform probability space:



Physical experiment



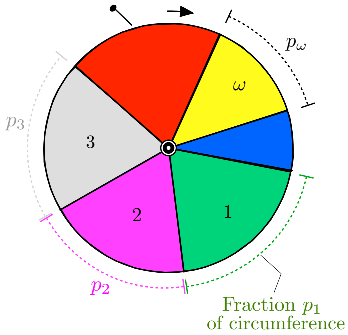
Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$
$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{ etc.}$$

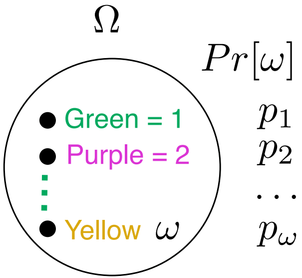
Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

Probability Space: Formalism

Physical model of a general non-uniform probability space:



Physical experiment



Probability model

The roulette wheel stops in sector ω with probability p_ω.

$$\Omega = \{1, 2, 3, \dots, N\}, Pr[\omega] = p_\omega.$$

An important remark

- The random experiment selects one and only one outcome in Ω.
- For instance, when we flip a fair coin twice
  - $\Omega = \{HH, TH, HT, TT\}$
  - The experiment selects *one* of the elements of Ω.
- In this case, its wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets HH or TT with probability 50% each. This is not captured by 'picking two outcomes.'

Example: Monty Hall Problem

Game Show: 3 doors.

1. Awesome prize behind one of the doors.
2. Contestant picks a door. Monty opens **one of the other** two doors that does not have a prize.
3. Contestant can either stick to the original door or change selection.

Is there any advantage to the contestant in switching?

Controversy

The problem appeared in "Parade" Magazine in a column by Marilyn Vos Savant.

From the Wikipedia

Though vos Savant gave the correct answer that switching would win two-thirds of the time, she estimates the magazine received 10,000 letters including close to 1,000 signed by PhDs, many on letterheads of mathematics and science departments, declaring that her solution was wrong (Tierney 1991). As a result of the publicity the problem earned the alternative name Marilyn and the Goats.

Monty Hall problem

- Two experiments, one in which he switches and one in which he doesn't.
- $\Omega_1 = \{$(Correct,Switch,Lose),(Wrong1,Switch,Win),(Wrong2,Switch,Win)$\}$ Prob of Winning is $\frac{2}{3}$.Another way: He only loses if picks the right door.
- $\Omega_2 = \{$(Correct,Stay,Win),(Wrong1,Stay,Lose),(Wrong2,Stay,Lose)$\}$.P of Winning is $\frac{1}{3}$.He only wins if he picks the right door.
- Much better to switch!

Why were so many professional mathematicians confused? They forgot about wrong door 2.

Summary

$\boxed{\text{Modeling Uncertainty: Probability Space}}$

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_{\omega} Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.

Probability problems can be fun

.

The n passengers of a full flight board sequentially. The first passenger sits in the wrong (unassigned) spot. Subsequent passengers either sit in their assigned seat or if it is not available, pick another seat at random. What is the the probability that the last passenger is able to sit in his assigned seat?