CS70: On to probability.

Modeling Uncertainty: Probability Space
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1. Key Points
2. Random Experiments
3. Probability Space
Key Points

▶ Uncertainty does not mean "nothing is known"
▶ Most real-world problems involve uncertainty
▶ Predictions:
  ▶ Will you get an A in CS70?
  ▶ Will the Raiders win the Super Bowl?
▶ Strategy/Decision-making under uncertainty
  ▶ Drop CS 70?
  ▶ How much to bet on blackjack?
  ▶ Buy a specific stock?
▶ Engineering
  ▶ Build a spam filter
  ▶ Improve wifi coverage.
  ▶ Control systems (Internet, airplane, robots, self-driving cars)
▶ How to best use 'artificial' uncertainty?
  ▶ Play games of chance
  ▶ Design randomized algorithms.
▶ Probability
  ▶ Models knowledge about uncertainty: Mathematical discipline that allows you to reason about uncertainty.
  ▶ Discovers best way to use that knowledge in making decisions
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The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability: a precise, unambiguous, simple(!) way to think about uncertainty.

Your mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice, practice, practice.
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Uncertainty: vague, fuzzy, confusing, scary, hard to think about.
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Uncertainty = Fear
The Magic of Probability

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Uncertainty = Fear

Probability = Serenity
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Random Experiment: Flip one Fair Coin

Flip a fair coin:

▶ Possible outcomes: Heads (H) and Tails (T)

▶ Likelihoods: H: 50% and T: 50%
Random Experiment: Flip one Fair Coin

Flip a fair coin:
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Flip a \textit{fair} coin: \textit{(One flips or tosses a coin)}
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

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Possible outcomes:
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads (*H*)
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Flip a fair coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads \((H)\) and Tails \((T)\)
Random Experiment: Flip one Fair Coin

Flip a fair coin: \((One \ flips \ or \ tosses \ a \ coin)\)

- Possible outcomes: Heads \((H)\) and Tails \((T)\)
  \((One \ flip \ yields \ either \ ‘heads’ \ or \ ‘tails’.\)\)
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Flip a **fair** coin: *(One flips or tosses a coin)*

- Possible outcomes: Heads *(H)* and Tails *(T)*
  *(One flip yields either ‘heads’ or ‘tails’).*

- Likelihoods:
Random Experiment: Flip one Fair Coin

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- Possible outcomes: Heads *(H)* and Tails *(T)*
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- Likelihoods: $H : 50\%$ and $T : 50\%$
Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:

▶ Single coin flip: 50% chance of 'tails' [subjectivist]

▶ Many coin flips: About half yield 'tails' [frequentist]

Makes sense for many flips

Question: Why does the fraction of tails converge to the same value every time? Statistical Regularity! Deep!
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Random Experiment: Flip one Fair Coin

Flip a \textit{fair} coin: model
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

[Diagram of a coin being flipped]

Physical Experiment

Probability Model

$\Omega = \{H, T\}$

$\Pr[H] = 0.5$

$\Pr[T] = 0.5$
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

The physical experiment is complex.
Random Experiment: Flip one Fair Coin

Flip a **fair** coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)

**Physical Experiment**

**Probability Model**
Random Experiment: Flip one Fair Coin

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- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

- The physical experiment is complex. (Shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set $\Omega$ of outcomes: $\Omega = \{H, T\}$. 
Random Experiment: Flip one Fair Coin

Flip a fair coin: model

The physical experiment is complex. (Shape, density, initial momentum and position, ...)

The Probability model is simple:

- A set \( \Omega \) of outcomes: \( \Omega = \{H, T\} \).
- A probability assigned to each outcome:
  \[ \Pr[H] = 0.5, \Pr[T] = 0.5. \]
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

Possible outcomes: Heads (H) and Tails (T)

Likelihoods:
H: p ∈ (0, 1) and T: 1 - p

Frequentist Interpretation:
Flip many times ⇒ Fraction 1 - p of tails

Question: How can one figure out p?

Flip many times

Tautology? No: Statistical regularity!
Random Experiment: Flip one Unfair Coin

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- Question: How can one figure out \( p \)?

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Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin:

- Possible outcomes:
  - Heads (H): 45%
  - Tails (T): 55%

Frequentist Interpretation: Flip many times ⇒ Fraction $1 - p$ of tails

Question: How can one figure out $p$?

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Flip an *unfair* (biased, loaded) coin:

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Flip an unfair (biased, loaded) coin: model
Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

Physical Experiment

Probability Model

\[ \Omega \]

\[ H \odot p \]

\[ T \odot (1 - p) \]
Flip Two Fair Coins

Possible outcomes: \{HH, HT, TH, TT\} ≡ \{H, T\}^2.

Note: \( A \times B := \{ (a, b) | a \in A, b \in B \} \) and \( A^2 := A \times A \).

Likelihoods: \( \frac{1}{4} \) each.
Flip Two Fair Coins

- Possible outcomes:
Flip Two Fair Coins

Possible outcomes: \{HH, HT, TH, TT\}
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- Possible outcomes: \( \{HH, HT, TH, TT\} \equiv \{H, T\}^2 \).
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- Likelihoods: 1/4 each.
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: {HH, TT}.  
- Likelihoods: HH: \(0.5\), TT: \(0.5\).

Note: Coins are glued so that they show the same face.
Flip Glued Coins

Flips two coins glued together side by side:
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \{HH, TT\}.

Likelihoods: HH: 0.5, TT: 0.5.

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Flips two coins glued together side by side:

- Possible outcomes:
  - HH: 0.5
  - TT: 0.5

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Possible outcomes: \{HH, TT\}. 
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- Likelihoods:
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \{HH, TT\}.

Likelihoods: \(HH : 0.5\), \(TT : 0.5\).
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Flips two coins glued together side by side:

- Possible outcomes: \{HH, TT\}.
- Likelihoods: HH : 0.5, TT : 0.5.
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Flip Glued Coins

Possible outcomes: {HT, TH}.

Likelihoods:
- HT: 0.5,
- TH: 0.5.

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \{HT, TH\}.

Likelihoods: HT: 0.5, TH: 0.5.

Note: Coins are glued so that they show different faces.
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes:
Flip Glued Coins

Flips two coins glued together side by side:

Possible outcomes: \( \{HT, TH\} \).
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Flips two coins glued together side by side:

- Possible outcomes: \{HT, TH\}.
- Likelihoods:
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \( \{HT, TH\} \).
- Likelihoods: \( HT : 0.5, TH : 0.5 \).
Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: \( \{HT, TH\} \).
- Likelihoods: \( HT : 0.5, TH : 0.5 \).
- Note: Coins are glued so that they show different faces.
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: HH, HT, TH, TT.
- Likelihoods:
  - HH: 0.4,
  - HT: 0.1,
  - TH: 0.1,
  - TT: 0.4.

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:
Flip two Attached Coins

Flips two coins attached by a spring:
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes:
  - HH: 0.4
  - HT: 0.1
  - TH: 0.1
  - TT: 0.4

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: \{HH, HT, TH, TT\}.
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: \( \{HH, HT, TH, TT\} \).
- Likelihoods:
Flip two Attached Coins

Flips two coins attached by a spring:

- Possible outcomes: \{HH, HT, TH, TT\}.
- Likelihoods: HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4.
Flip two Attached Coins

Flips two coins attached by a spring:

Possible outcomes: \{HH, HT, TH, TT\}.

Likelihoods: \(HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4\).

Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
Flipping Two Coins

Here is a way to summarize the four random experiments:

- $\Omega$ is the set of possible outcomes;
- Each outcome has a probability (likelihood);
- The probabilities are $\geq 0$ and add up to 1;
Flipping Two Coins

Here is a way to summarize the four random experiments:
Flipping Two Coins

Here is a way to summarize the four random experiments:

1. Fair coins: $\Omega = \{TH, HH, TT, HT\}$
   - $TH = 0.25$, $HH = 0.25$, $TT = 0.25$, $HT = 0.25$

2. Glued coins: $\Omega = \{TH, HH, TT, HT\}$
   - $TH = 0.1$, $HH = 0.4$, $TT = 0.4$, $HT = 0.1$

3. Spring-attached coins: $\Omega = \{TH, HH, TT, HT\}$
   - $TH = 0$, $HH = 0.5$, $TT = 0.5$, $HT = 0$

4. Glued coins: $\Omega = \{TH, HH, TT, HT\}$
   - $TH = 0.5$, $HH = 0$, $TT = 0$, $HT = 0.5$
Flipping Two Coins

Here is a way to summarize the four random experiments:

- $\Omega$ is the set of possible outcomes;

\[\begin{array}{c}
\text{[1]} & \text{[2]} & \text{[3]} & \text{[4]}
\end{array}\]

- Fair coins: [\( TH \) 0.25 0.25; \( TT \) 0.25 0.25];
- Glued coins: [\( TH \) 0.1 0.4; \( TT \) 0.4 0.1]; [\( TH \) 0 0.5; \( TT \) 0.5 0];
- Spring-attached coins: [\( TH \) 0.5 0; \( TT \) 0 0.5].
Here is a way to summarize the four random experiments:

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Here is a way to summarize the four random experiments:

1. \( \Omega \) is the set of possible outcomes;
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Important remarks:
Flipping Two Coins

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- Each outcome describes the two coins.

![Diagram showing different outcomes and their probabilities for flipping two coins.](image)
Flipping Two Coins

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▶ Each outcome describes the two coins.
▶ E.g., $HT$ is one outcome of the experiment.
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- Indeed, this viewpoint misses the relationship between the two flips.
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- It is wrong to think that the outcomes are \{H, T\} and that one picks twice from that set.
- Indeed, this viewpoint misses the relationship between the two flips.
- Each \(\omega \in \Omega\) describes one outcome of the complete experiment.
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► Indeed, this viewpoint misses the relationship between the two flips.
► Each $\omega \in \Omega$ describes one outcome of the complete experiment.
► $\Omega$ and the probabilities specify the random experiment.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):
Flipping $n$ times

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- Possible outcomes:
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$. 

Thus, $2^n$ possible outcomes.

Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H_1, T\}^n$.

$A_n := \{(a_1, \ldots, a_n) | a_1 \in A, \ldots, a_n \in A\}$.

$|A_n| = |A|^n$. 

Likelihoods: $1/2^n$ each.
Flipping $n$ times

Flip a fair coin $n$ times (some $n \geq 1$):

- Possible outcomes: \{TT \ldots T, TT \ldots H, \ldots, HH\ldots H\}.
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Flipping \( n \) times

Flip a **fair** coin \( n \) times (some \( n \geq 1 \)):

- Possible outcomes: \( \{ TT \ldots T, TT \ldots H, \ldots, HH \ldots H \} \).
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- Note: \( \{ TT \ldots T, TT \ldots H, \ldots, HH \ldots H \} = \{ H, T \}^n \).

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Roll two Dice

Roll a balanced 6-sided die twice:
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes:
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes: \( \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\} \).
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes: \( \{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\} \).
- Likelihoods:
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes: $\{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\}$.
- Likelihoods: $1/36$ for each.
Roll two Dice

Roll a balanced 6-sided die twice:

- Possible outcomes: $\{1, 2, 3, 4, 5, 6\}^2 = \{(a, b) \mid 1 \leq a, b \leq 6\}$.
- Likelihoods: $1/36$ for each.
Probability Space.

1. A “random experiment”: 

(a) Flip a biased coin; 
(b) Flip two fair coins; 
(c) Deal a poker hand.
Probability Space.

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   (a) \( \Omega = \{H, T\} \);
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1. A “random experiment”:
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   (a) $\Omega = \{H, T\}$;
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2. A set of possible outcomes: $\Omega$.
   (a) $\Omega = \{H, T\}$;
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Probability Space.

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   (a) Flip a biased coin;
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2. A set of possible outcomes: $\Omega$.
   (a) $\Omega = \{H, T\}$;
   (b) $\Omega = \{HH, HT, TH, TT\}; |\Omega| = 4$;
   (c) $\Omega = \{A\spadesuit A\diamondsuit A\clubsuit A\heartsuit K\spadesuit, A\spadesuit A\diamondsuit A\clubsuit A\heartsuit Q\spadesuit, \ldots\}$
      $|\Omega| =$
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   (a) $\Omega = \{H, T\}$;
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   $|\Omega| = \binom{52}{5}$. 
Probability Space.

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   (a) $\Omega = \{H, T\}$;
   (b) $\Omega = \{HH, HT, TH, TT\}$; $|\Omega| = 4$;
   (c) $\Omega = \{ A♠ A♦ A♣ A♥ K♠, A♠ A♦ A♣ A♥ Q♠, \ldots \}$
      $|\Omega| = \binom{52}{5}$.

3. Assign a probability to each outcome: $Pr : \Omega \rightarrow [0, 1]$.
   (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0, 1]$
Probability Space.

1. A “random experiment”:
   (a) Flip a biased coin;
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2. A set of possible outcomes: $\Omega$.
   (a) $\Omega = \{H, T\}$;
   (b) $\Omega = \{HH, HT, TH, TT\}; |\Omega| = 4$;
   (c) $\Omega = \{A\spadesuit A\Diamond A\spadesuit A\heartsuit K\spadesuit, A\spadesuit A\Diamond A\spadesuit A\heartsuit Q\spadesuit, \ldots\}$
      $|\Omega| = \binom{52}{5}$.

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   (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0, 1]$
   (b) $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$
Probability Space.

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   (a) Flip a biased coin;
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   (a) \( \Omega = \{ H, T \} \);
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   (c) \( \Omega = \{ A\spadesuit A\diamondsuit A\heartsuit A\clubsuit K\spadesuit, A\spadesuit A\diamondsuit A\clubsuit A\heartsuit Q\spadesuit, \ldots \} \)
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   (a) \( Pr[H] = p, Pr[T] = 1 - p \) for some \( p \in [0, 1] \)
   (b) \( Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4} \)
   (c) \( Pr[ A\spadesuit A\diamondsuit A\clubsuit A\heartsuit K\spadesuit ] = \cdots = \frac{1}{\binom{52}{5}} \)
Probability Space: formalism.

\( \Omega \) is the **sample space**.
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Ω is the **sample space**.
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Sample point ω has a probability Pr[ω] where
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\[ 0 \leq Pr[\omega] \leq 1; \]
Probability Space: formalism.

$\Omega$ is the **sample space**.  
$\omega \in \Omega$ is a **sample point**. (Also called an **outcome**.) 
Sample point $\omega$ has a probability $Pr[\omega]$ where

- $0 \leq Pr[\omega] \leq 1$;
- $\sum_{\omega \in \Omega} Pr[\omega] = 1$. 
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Ω is the **sample space**.
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Sample point ω has a probability $Pr[ω]$ where

- $0 ≤ Pr[ω] ≤ 1;$
- $\sum_{ω \in Ω} Pr[ω] = 1$. 

![Diagram of probability space](image)
Probability Space: Formalism.

In a **uniform probability space** each outcome $\omega$ is **equally probable**:

$$Pr[\omega] = \frac{1}{|\Omega|} \text{ for all } \omega \in \Omega.$$
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**Examples:**

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
In a **uniform probability space** each outcome $\omega$ is equally probable: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.

Examples:

- Flipping two fair coins, dealing a poker hand are uniform probability spaces.
- Flipping a biased coin is not a uniform probability space.
Probability Space: Formalism

Simplest physical model of a uniform probability space:

A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

Ω = \{white, red, yellow, grey, purple, blue, maroon, green\}

Pr[blue] = \frac{1}{8}.
Probability Space: Formalism

Simplest physical model of a uniform probability space:

Physical experiment

Probability model

\[ \Omega \]

\[ Pr[\omega] \]

- Red
  \[ 1/8 \]
- Green
  \[ 1/8 \]
- Maroon
  \[ 1/8 \]
- ...
Probability Space: Formalism

Simplest physical model of a **uniform** probability space:

A bag of identical balls, except for their color (or a label).
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A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

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Probability Space: Formalism

Simplest physical model of a non-uniform probability space:
Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ \Pr[\omega] = \begin{cases} 
3/10 & \text{Red} \\
4/10 & \text{Green} \\
2/10 & \text{Yellow} \\
1/10 & \text{Blue} 
\end{cases} \]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[
\begin{align*}
\Omega &= \{\text{Red, Green, Yellow, Blue}\} \\
& \quad \text{Physical experiment} \\
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
\text{Red} \\
\text{Green} \\
\text{Yellow} \\
\text{Blue}
\end{bmatrix}
& \quad \text{Probability model} \\
& \quad \begin{bmatrix}
\frac{3}{10} \\
\frac{4}{10} \\
\frac{2}{10} \\
\frac{1}{10}
\end{bmatrix}
\end{align*}
\]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10} \]

\[ Pr[\text{Green}] = \frac{4}{10} \]

\[ Pr[\text{Yellow}] = \frac{2}{10} \]

\[ Pr[\text{Blue}] = \frac{1}{10} \]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

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\[ Pr[\text{Green}] = \frac{4}{10}, \]

\[ Pr[\text{Yellow}] = \frac{2}{10}, \]

\[ Pr[\text{Blue}] = \frac{1}{10} \]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10}, \quad Pr[\text{Green}] = \frac{4}{10}, \quad Pr[\text{Yellow}] = \frac{2}{10}, \quad Pr[\text{Blue}] = \frac{1}{10} \]
Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]
\[ Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{etc.} \]
Simplest physical model of a non-uniform probability space:

\[ \Omega = \{ \text{Red, Green, Yellow, Blue} \} \]

\[ Pr[\text{Red}] = \frac{3}{10}, \quad Pr[\text{Green}] = \frac{4}{10}, \quad \text{etc.} \]

Note: Probabilities are restricted to rational numbers: \( \frac{N_k}{N} \).
Probability Space: Formalism

Physical model of a general non-uniform probability space:
Probability Space: Formalism

Physical model of a general non-uniform probability space:

Physical experiment

Probability model

\[ \Omega = \{1, 2, 3, \ldots, N\} \]

\[ Pr[\omega] = p_\omega \]

- Green = 1
- Purple = 2
- Yellow

Fraction \( p_1 \) of circumference

\( p_2 \)

\( p_3 \)
Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$. 
Probability Space: Formalism

Physical model of a general non-uniform probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$$\Omega = \{1, 2, 3, \ldots, N\},$$
Probability Space: Formalism

Physical model of a general **non-uniform** probability space:

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$$\Omega = \{1, 2, 3, \ldots, N\}, Pr[\omega] = p_\omega.$$
An important remark

- The random experiment selects one and only one outcome in $\Omega$. 
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An important remark

- The random experiment selects **one and only one** outcome in $\Omega$.
- For instance, when we flip a fair coin **twice**
  - $\Omega = \{HH, TH, HT, TT\}$
  - The experiment selects **one** of the elements of $\Omega$. 
An important remark

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- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets HH or TT with probability 50% each. This is not captured by ‘picking two outcomes.’
Example: Monty Hall Problem

Game Show: 3 doors.

1. Awesome prize behind one of the doors.
2. Contestant picks a door. Monty opens one of the other two doors that does not have a prize.
3. Contestant can either stick to the original door or change selection.

Is there any advantage to the contestant in switching?
The problem appeared in "Parade" Magazine in a column by Marilyn Vos Savant.

From the Wikipedia
Though vos Savant gave the correct answer that switching would win two-thirds of the time, she estimates the magazine received 10,000 letters including close to 1,000 signed by PhDs, many on letterheads of mathematics and science departments, declaring that her solution was wrong (Tierney 1991). As a result of the publicity the problem earned the alternative name Marilyn and the Goats.
Two experiments, one in which he switches and one in which he doesn’t.

\[ \Omega_1 = \{(\text{Correct, Switch, Lose}), (\text{Wrong1, Switch, Win}), (\text{Wrong2, Switch, Win})\} \]

Prob of Winning is \( \frac{2}{3} \). Another way: He only loses if picks the right door.

\[ \Omega_2 = \{(\text{Correct, Stay, Win}), (\text{Wrong1, Stay, Lose}), (\text{Wrong2, Stay, Lose})\} \]

Prob of Winning is \( \frac{1}{3} \). He only wins if he picks the right door.

Much better to switch!

Why were so many professional mathematicians confused? They forgot about wrong door 2.
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1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0, 1]; \sum_\omega Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$. 
The \( n \) passengers of a full flight board sequentially. The first passenger sits in the wrong (unassigned) spot. Subsequent passengers either sit in their assigned seat or if it is not available, pick another seat at random. What is the probability that the last passenger is able to sit in his assigned seat?