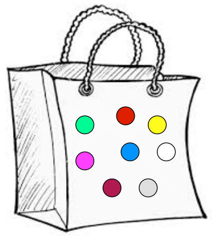
## Continuing Probability.

Wrap up: Probability Formalism.

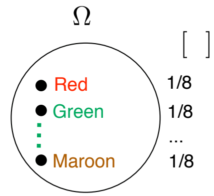
Events, Conditional Probability, Independence, Bayes' Rule

## Probability Space: Formalism

Simplest physical model of a uniform probability space:



Physical experiment




Probability model

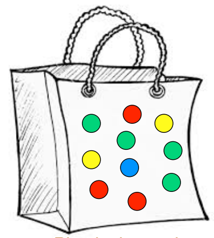
A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$

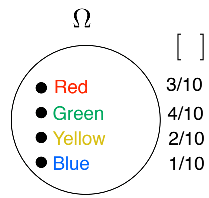
$$Pr[\text{blue}] = \frac{1}{8}.$$

## Probability Space: Formalism

Simplest physical model of a non-uniform probability space:



Physical experiment




Probability model

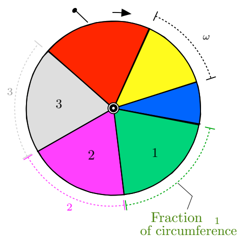
$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{ etc.}$$

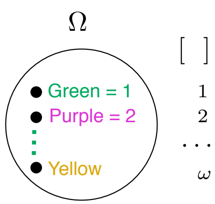
Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

## Probability Space: Formalism

Physical model of a general non-uniform probability space:




Physical experiment




Probability model

The roulette wheel stops in sector $\omega$ with probability $p_\omega$.

$$\Omega = \{1,2,3,\ldots,N\}, Pr[\omega] = p_\omega.$$

## An important remark

- The random experiment selects one and only one outcome in $\Omega$.
- For instance, when we flip a fair coin twice
  - $\Omega = \{HH, TH, HT, TT\}$
  - The experiment selects *one* of the elements of $\Omega$.
- In this case, its wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.
- Why? Because this would not describe how the two coin flips are related to each other.
- For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets $HH$ or $TT$ with probability 50% each. This is not captured by 'picking two outcomes.'

## Lecture 15: Summary

Modeling Uncertainty: Probability Space

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_\omega Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.

## CS70: On to Calculation.

Events, Conditional Probability, Independence, Bayes' Rule

1. Probability Basics Review
2. Events
3. Conditional Probability
4. Independence of Events
5. Bayes' Rule

## Probability Basics Review

**Setup:**

- Random Experiment.
  Flip a fair coin twice.
- Probability Space.
  - **Sample Space:** Set of outcomes, $\Omega$.
    $\Omega = \{HH, HT, TH, TT\}$
    (Note: Not $\Omega = \{H, T\}$ with two picks!)
  - **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \le Pr[\omega] \le 1$.
    2. $\sum_{\omega \in \Omega} Pr[\omega] = 1$.

## Set notation review

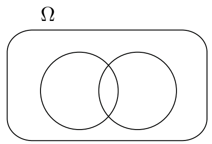



Figure: Two events

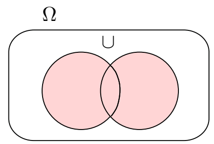



Figure: Union (or)

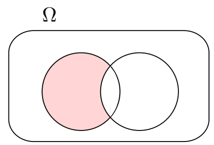



Figure: Difference ($A$, not $B$)

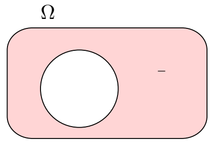



Figure: Complement (not)

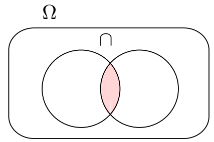



Figure: Intersection (and)

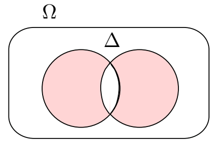



Figure: Symmetric difference (only one)

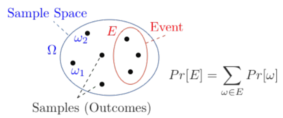
## Probability of exactly one 'heads' in two coin flips?

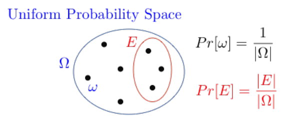
Idea: Sum the probabilities of all the different outcomes that have exactly one 'heads': $HT, TH$.

This leads to a definition!

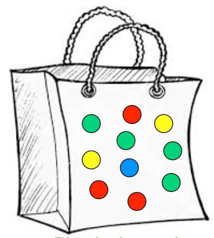
**Definition:**

- An **event,** $E$**,** is a subset of outcomes: $E \subset \Omega$.
- The **probability of** $E$ is defined as $Pr[E] = \sum_{\omega \in E} Pr[\omega]$.

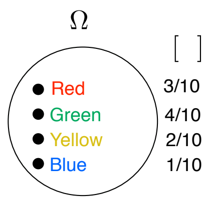



## Event: Example



Physical experiment




Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{ etc.}$$

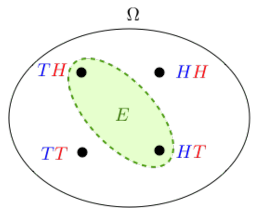
$$E = \{Red, Green\} \Rightarrow Pr[E] = \frac{3+4}{10} = \frac{3}{10} + \frac{4}{10} = Pr[\text{Red}] + Pr[\text{Green}].$$

## Probability of exactly one heads in two coin flips?

Sample Space, $\Omega = \{HH, HT, TH, TT\}$.

Uniform probability space:
$Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$.

Event, $E$, "exactly one heads": $\{TH, HT\}$.




$$Pr[E] = \sum_{\omega \in E} Pr[\omega] = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}.$$

## Example: 20 coin tosses.

**20 coin tosses**

Sample space: $\Omega =$ set of 20 fair coin tosses.
$\Omega = \{T,H\}^{20} \equiv \{0,1\}^{20}$; $|\Omega| = 2^{20}$.

- What is more likely?
  - $\omega_1 := (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$, or
  - $\omega_2 := (1,0,1,1,0,0,0,1,0,1,0,1,1,0,1,1,1,0,0,0)$?

  Answer: Both are equally likely: $Pr[\omega_1] = Pr[\omega_2] = \frac{1}{|\Omega|}$.
- What is more likely?

  ($E_1$) Twenty Hs out of twenty, or
  ($E_2$) Ten Hs out of twenty?

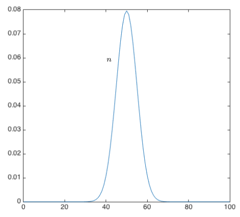
  Answer: Ten Hs out of twenty.

  Why? There are many sequences of 20 tosses with ten Hs; only one with twenty Hs. $\Rightarrow Pr[E_1] = \frac{1}{|\Omega|} \ll Pr[E_2] = \frac{|E_2|}{|\Omega|}$.

$$|E_2| = \binom{20}{10} = 184,756.$$

## Probability of $n$ heads in 100 coin tosses.

$\Omega = \{H,T\}^{100}$; $|\Omega| = 2^{100}$.



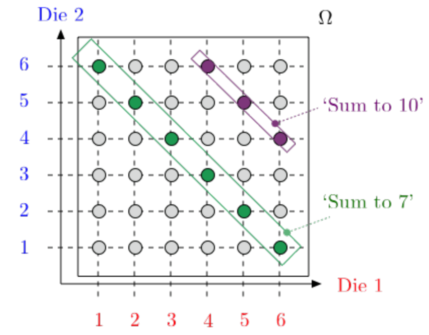

Event $E_n$ = '$n$ heads'; $|E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

Observe:

- Concentration around mean: Law of Large Numbers;
- Bell-shape: Central Limit Theorem.

## Roll a red and a blue die.




$Pr[\text{Sum to 7}] = \frac{6}{36}$ $\quad Pr[\text{Sum to 10}] = \frac{3}{36}$

## Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H,T\}^{100}$.
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$

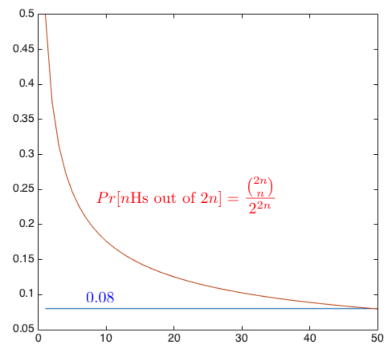
**Calculation.**
Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n}(2n/e)^{2n}}{[\sqrt{2\pi n}(n/e)^n]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$

## Exactly 50 heads in 100 coin tosses.



## Probability is Additive

**Theorem**

(a) If events $A$ and $B$ are disjoint, i.e., $A \cap B = \emptyset$, then

$$Pr[A \cup B] = Pr[A] + Pr[B].$$

(b) If events $A_1, \ldots, A_n$ are pairwise disjoint,
i.e., $A_k \cap A_m = \emptyset, \forall k \neq m$, then

$$Pr[A_1 \cup \cdots \cup A_n] = Pr[A_1] + \cdots + Pr[A_n].$$

**Proof:**

Obvious.

## Consequences of Additivity

**Theorem**

(a) $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$;
(inclusion-exclusion property)

(b) $Pr[A_1 \cup \cdots \cup A_n] \leq Pr[A_1] + \cdots + Pr[A_n]$;
(union bound)

(c) If $A_1, \ldots A_N$ are a partition of $\Omega$, i.e.,
pairwise disjoint and $\cup_{m=1}^{N} A_m = \Omega$, then

$$Pr[B] = Pr[B \cap A_1] + \cdots + Pr[B \cap A_N].$$

(law of total probability)

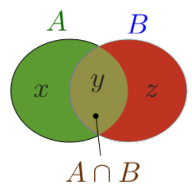
**Proof:**

(b) is obvious.

Proofs for (a) and (c)? Next...

## Inclusion/Exclusion

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

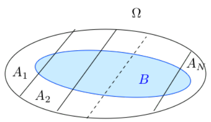



$$Pr[A] = x + y$$
$$Pr[B] = y + z$$
$$Pr[A \cap B] = y$$
$$Pr[A \cup B] = x + y + z$$

Another view. Any $\omega \in A \cup B$ is in $A \cap \overline{B}$, $A \cup B$, or $\overline{A} \cap B$. So, add it up.

## Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.




Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$.

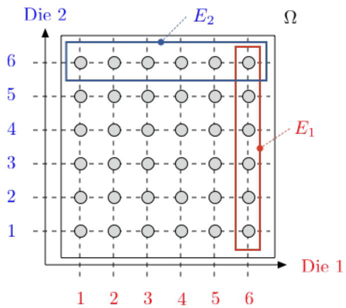
In "math": $\omega \in B$ is in exactly one of $A_i \cap B$.

Adding up probability of them, get $Pr[\omega]$ in sum.

..Did I say...

Add it up.

## Roll a Red and a Blue Die.



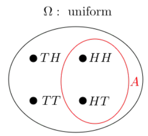

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$E_1 =$ 'Red die shows 6'; $E_2 =$ 'Blue die shows 6'
$E_1 \cup E_2 =$ 'At least one die shows 6'

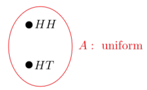
$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$

## Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?
$\Omega = \{HH, HT, TH, TT\}$; Uniform probability space.
Event $A =$ first flip is heads: $A = \{HH, HT\}$.




New sample space: $A$; uniform still.



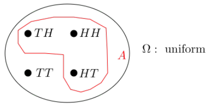

Event $B =$ two heads.

The probability of two heads if the first flip is heads.
**The probability of $B$ given $A$ is** $1/2$.

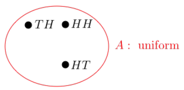
## A similar example.

Two coin flips. At least one of the flips is heads.
$\rightarrow$ Probability of two heads?

$\Omega = \{HH, HT, TH, TT\}$; uniform.
Event $A =$ at least one flip is heads. $A = \{HH, HT, TH\}$.



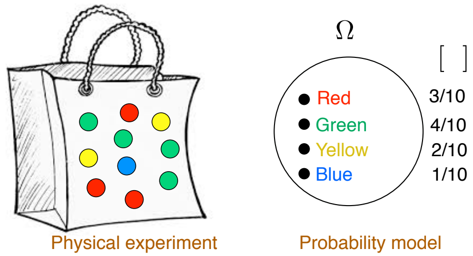

New sample space: $A$; uniform still.




Event $B =$ two heads.

The probability of two heads if at least one flip is heads.
**The probability of** $B$ **given** $A$ is $1/3$.

## Conditional Probability: A non-uniform example

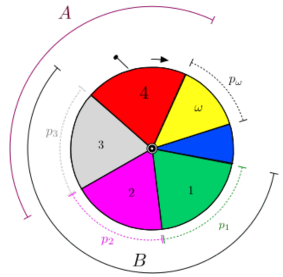



$\Omega = \{\text{Red, Green, Yellow, Blue}\}$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

## Another non-uniform example

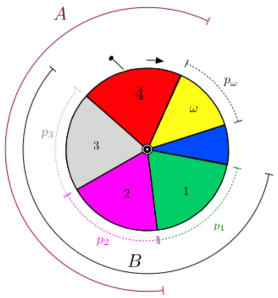
Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.
Let $A = \{3, 4\}, B = \{1, 2, 3\}$.




$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

## Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.
Let $A = \{2, 3, 4\}, B = \{1, 2, 3\}$.

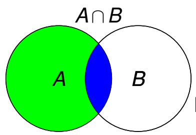



$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

## Conditional Probability.

**Definition:** The **conditional probability** of $B$ given $A$ is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$

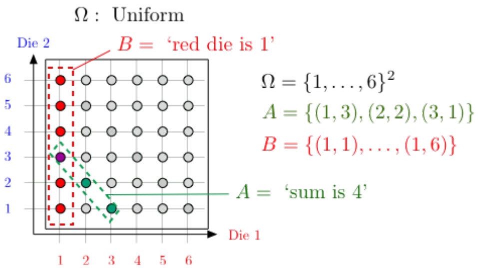



In $A$!
In $B$?
Must be in $A \cap B$.

$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$.

## More fun with conditional probability.

Toss a red and a blue die, sum is 4,
What is probability that red is 1?

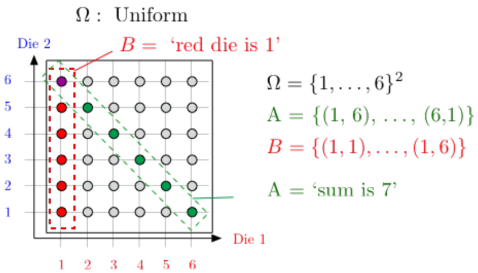



$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{3}$; versus $Pr[B] = 1/6$.

$B$ is more likely given $A$.

## Yet more fun with conditional probability.

Toss a red and a blue die, sum is 7,
what is probability that red is 1?

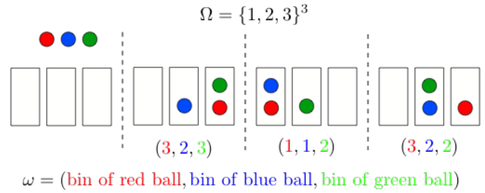



$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{6}$; versus $Pr[B] = \frac{1}{6}$.

Observing $A$ does not change your mind about the likelihood of $B$.

## Emptiness..

Suppose I toss 3 balls into 3 bins.
$A$ ="1st bin empty"; $B$ ="2nd bin empty." What is $Pr[A|B]$?




$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] = Pr[\{1,3\}^3] = \frac{8}{27}$

$Pr[A \cap B] = Pr[(3,3,3)] = \frac{1}{27}$

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{(1/27)}{(8/27)} = 1/8$; vs. $Pr[A] = \frac{8}{27}$.

$A$ is less likely given $B$: If second bin is empty the first is more likely to have balls in it.

## Three Card Problem

Three cards: Red/Red, Red/Black, Black/Black.
Pick one at random and place on the table. The upturned side is a Red. What is the probability that the other side is Black?
Can't be the BB card, so...prob should be 0.5, right?
$R$: upturned card is Red; $RB$: the Red/Black card was selected.
Want $P(RB|R)$.
What's wrong with the reasoning that leads to $\frac{1}{2}$?

$$\begin{aligned} P(RB|R) &= \frac{P(RB \cap R)}{P(R)} \\ &= \frac{\frac{1}{3}\frac{1}{2}}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}(0)} \\ &= \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \end{aligned}$$

Once you are given $R$: it is twice as likely that the RR card was picked.



## Gambler's fallacy.

Flip a fair coin 51 times.
$A$ = "first 50 flips are heads"
$B$ = "the 51st is heads"
$Pr[B|A]$ ?

$A = \{HH \cdots HT, HH \cdots HH\}$
$B \cap A = \{HH \cdots HH\}$

Uniform probability space.

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{2}$.

Same as $Pr[B]$.

The likelihood of 51st heads does not depend on the previous flips.

## Product Rule

Recall the definition:

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$

Hence,

$$Pr[A \cap B] = Pr[A]Pr[B|A].$$

Consequently,

$$\begin{aligned} Pr[A \cap B \cap C] &= Pr[(A \cap B) \cap C] \\ &= Pr[A \cap B]Pr[C|A \cap B] \\ &= Pr[A]Pr[B|A]Pr[C|A \cap B]. \end{aligned}$$

## Product Rule

**Theorem** Product Rule
Let $A_1, A_2, \ldots, A_n$ be events. Then

$$Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1]Pr[A_2|A_1] \cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

**Proof:** By induction.
Assume the result is true for $n$. (It holds for $n = 2$.) Then,

$$\begin{aligned} & Pr[A_1 \cap \cdots \cap A_n \cap A_{n+1}] \\ & = Pr[A_1 \cap \cdots \cap A_n]Pr[A_{n+1}|A_1 \cap \cdots \cap A_n] \\ & = Pr[A_1]Pr[A_2|A_1] \cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}]Pr[A_{n+1}|A_1 \cap \cdots \cap A_n], \end{aligned}$$

so that the result holds for $n+1$. $\square$

## Correlation

An example.
Random experiment: Pick a person at random.
Event $A$: the person has lung cancer.
Event $B$: the person is a heavy smoker.

Fact:

$$Pr[A|B] = 1.17 \times Pr[A].$$

Conclusion:

- Smoking increases the probability of lung cancer by 17%.
- Smoking causes lung cancer.

## Correlation

Event $A$: the person has lung cancer. Event $B$: the person is a heavy smoker. $Pr[A|B] = 1.17 \times Pr[A]$.

A second look.

Note that

$$\begin{aligned} Pr[A|B] = 1.17 \times Pr[A] &\Leftrightarrow \frac{Pr[A \cap B]}{Pr[B]} = 1.17 \times Pr[A] \\ &\Leftrightarrow Pr[A \cap B] = 1.17 \times Pr[A]Pr[B] \\ &\Leftrightarrow Pr[B|A] = 1.17 \times Pr[B]. \end{aligned}$$

Conclusion:

- Lung cancer increases the probability of smoking by 17%.
- Lung cancer causes smoking. Really?

## Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

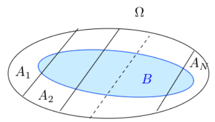
$A$ and $B$ being positively correlated does not mean that $A$ causes $B$ or that $B$ causes $A$.

Other examples:

- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?

## Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \dots, A_N$.




Then,

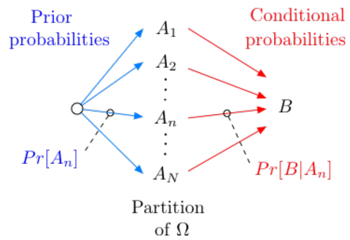
$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \dots, N$. Thus,

$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_N]Pr[B|A_N].$$

## Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \dots, A_N$.




$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_N]Pr[B|A_N].$$

## Is your coin loaded?

Your coin is fair w.p. $1/2$ or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

**Analysis:**

$$A = \text{'coin is fair'}, B = \text{'outcome is heads'}$$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$

Now,

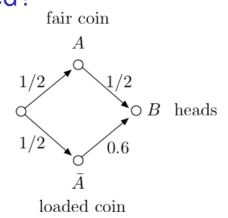
$$\begin{aligned} Pr[B] &= Pr[A \cap B] + Pr[\bar{A} \cap B] = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \\ &= (1/2)(1/2) + (1/2)0.6 = 0.55. \end{aligned}$$

Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

## Is your coin loaded?

A picture:




Imagine 100 situations, among which
$m := 100(1/2)(1/2)$ are such that $A$ and $B$ occur and
$n := 100(1/2)(0.6)$ are such that $\bar{A}$ and $B$ occur.

Thus, among the $m+n$ situations where $B$ occurred, there are $m$ where $A$ occurred.

Hence,

$$Pr[A|B] = \frac{m}{m+n} = \frac{(1/2)(1/2)}{(1/2)(1/2)+(1/2)0.6}.$$

## Independence

**Definition:** Two events $A$ and $B$ are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent;
- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent;
- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent;
- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are not independent;

## Independence and conditional probability

**Fact:** Two events $A$ and $B$ are **independent** if and only if

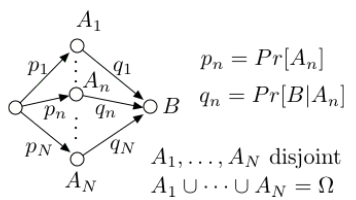
$$Pr[A|B] = Pr[A].$$

Indeed: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$, so that

$$Pr[A|B] = Pr[A] \Leftrightarrow \frac{Pr[A \cap B]}{Pr[B]} = Pr[A] \Leftrightarrow Pr[A \cap B] = Pr[A]Pr[B].$$

## Bayes Rule

Another picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$.

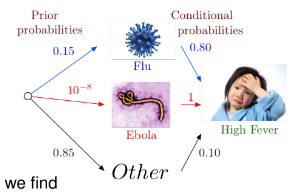



Imagine 100 situations, among which $100p_nq_n$ are such that $A_n$ and $B$ occur, for $n = 1, \ldots, N$.
Thus, among the $100\sum_m p_mq_m$ situations where $B$ occurred, there are $100p_nq_n$ where $A_n$ occurred.

Hence,

$$Pr[A_n|B] = \frac{p_nq_n}{\sum_m p_mq_m}.$$

## Why do you have a fever?




Using Bayes' rule, we find

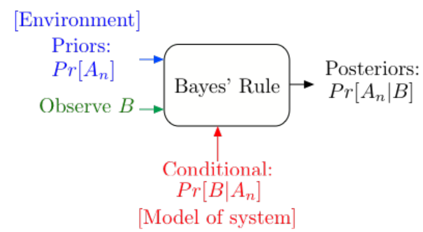
$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

$$Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}$$

$$Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42$$

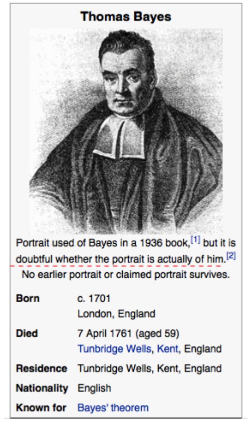
These are the posterior probabilities. One says that 'Flu' is the Most Likely a Posteriori (MAP) cause of the high fever.

## Bayes' Rule Operations




Bayes' Rule is the canonical example of how information changes our opinions.

## Thomas Bayes


**Thomas Bayes**

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

| | |
|---|---|
| **Born** | c. 1701<br>London, England |
| **Died** | 7 April 1761 (aged 59)<br>Tunbridge Wells, Kent, England |
| **Residence** | Tunbridge Wells, Kent, England |
| **Nationality** | English |
| **Known for** | Bayes' theorem |

Source: Wikipedia.

## Thomas Bayes

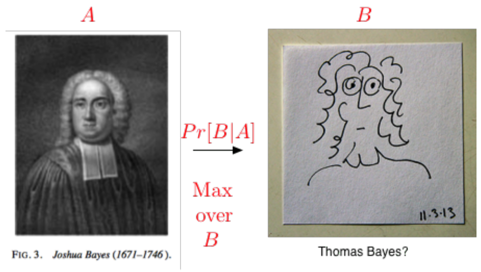


FIG. 3. *Joshua Bayes (1671–1746).*

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

## Testing for disease.

Let's watch TV!!
Random Experiment: Pick a random male.
Outcomes: $(test, disease)$
$A$ - prostate cancer.
$B$ - positive PSA test.

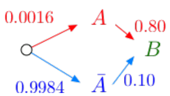
- $Pr[A] = 0.0016$, (.16 % of the male population is affected.)
- $Pr[B|A] = 0.80$ (80% chance of positive test with disease.)
- $Pr[B|\bar{A}] = 0.10$ (10% chance of positive test without disease.)

From http://www.cpcn.org/01_psa_tests.htm and http://seer.cancer.gov/statfacts/html/prost.html (10/12/2011.)

Positive PSA test ($B$). Do I have disease?

$$Pr[A|B]???$$

## Bayes Rule.




Using Bayes' rule, we find

$$P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = .013.$$

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence..

Death.

## Summary

$\boxed{\text{Events, Conditional Probability, Independence, Bayes' Rule}}$

Key Ideas:

- Conditional Probability:
  $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$
- Independence: $Pr[A \cap B] = Pr[A]Pr[B]$.
- Bayes' Rule:

$$Pr[A_n|B] = \frac{Pr[A_n]Pr[B|A_n]}{\sum_m Pr[A_m]Pr[B|A_m]}.$$

$Pr[A_n|B]$ = posterior probability; $Pr[A_n]$ = prior probability .

- All these are possible:
  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A]$.