Continuing Probability.

Wrap up: Probability Formalism.

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Events, Conditional Probability, Independence, Bayes' Rule

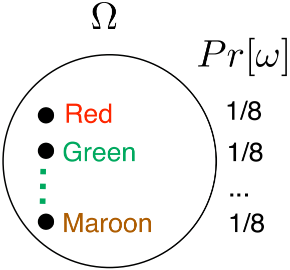
Probability Space: Formalism

Simplest physical model of a uniform probability space:

Probability Space: Formalism

Simplest physical model of a uniform probability space:

Physical experiment



Probability model

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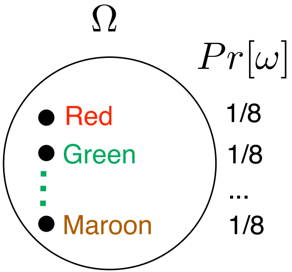

Probability model

A bag of identical balls, except for their color (or a label).

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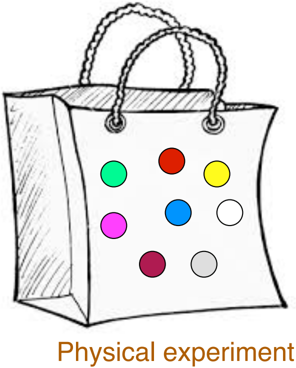

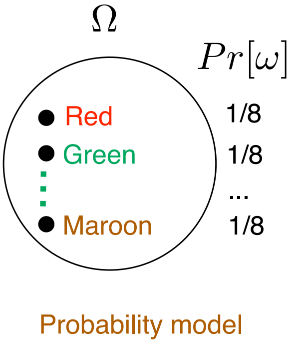

A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

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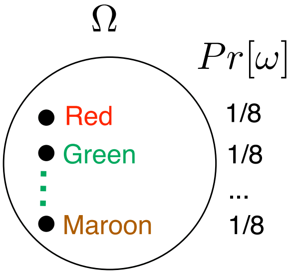

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$\Omega = \{$white, red, yellow, grey, purple, blue, maroon, green$\}$

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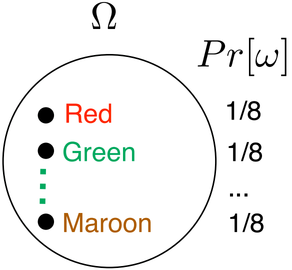

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$$Pr[\text{blue}] =$$

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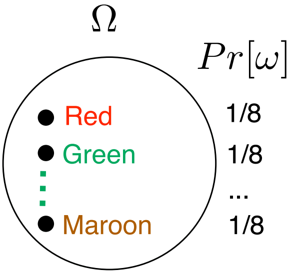

Probability model

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$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$

$$Pr[\text{blue}] = \frac{1}{8}.$$

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Simplest physical model of a non-uniform probability space:

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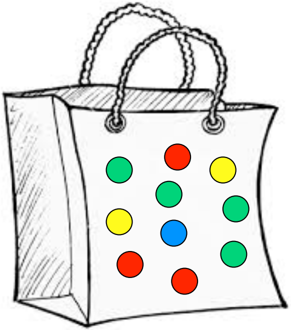
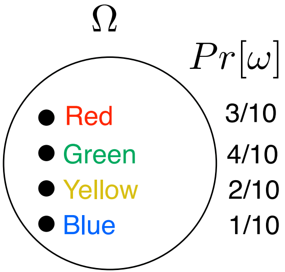


Physical experiment



Probability model

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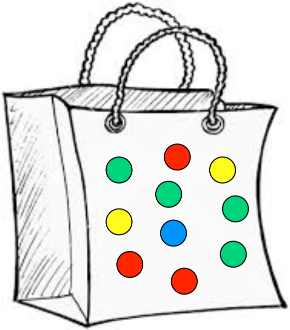

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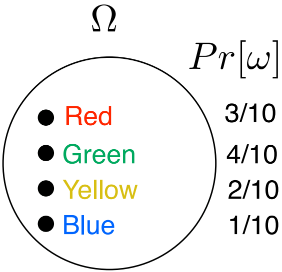

Probability model

$\Omega = \{\text{Red, Green, Yellow, Blue}\}$

Probability Space: Formalism

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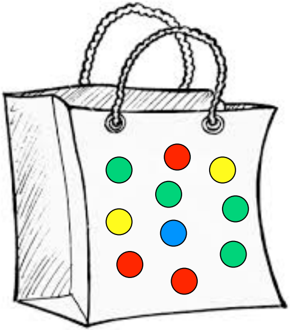

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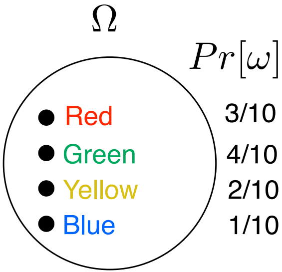

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] =$$

Probability Space: Formalism

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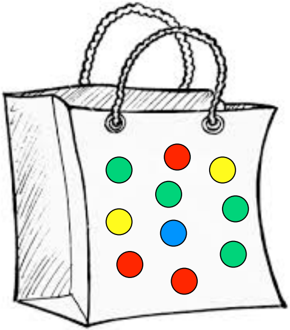

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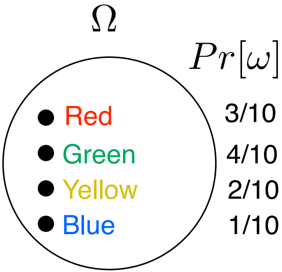

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}] = \frac{3}{10},$$

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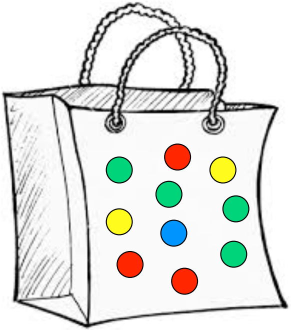

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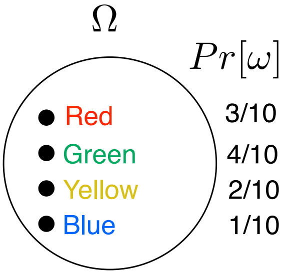

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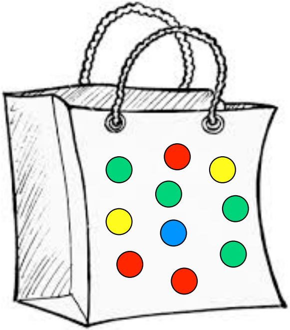

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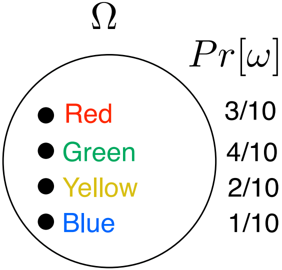

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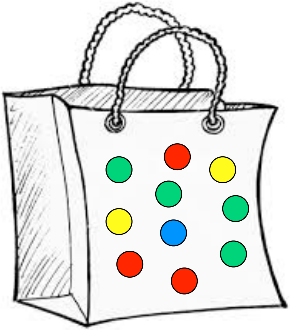

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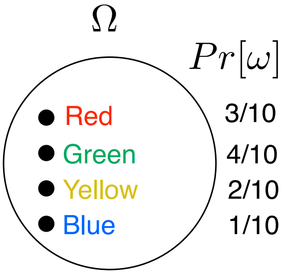

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Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

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Physical model of a general non-uniform probability space:

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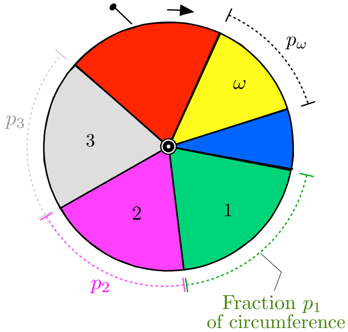

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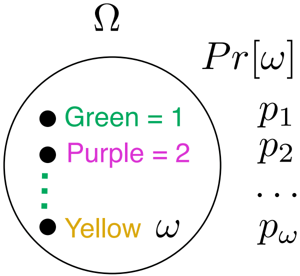

Probability model

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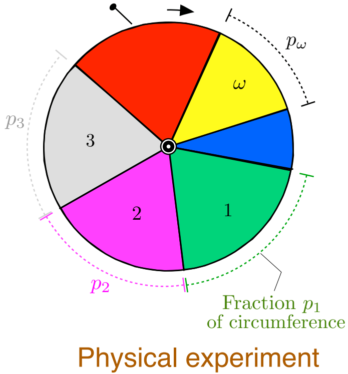

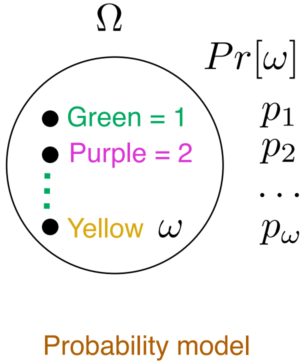

The roulette wheel stops in sector ω with probability p_ω.

Probability Space: Formalism

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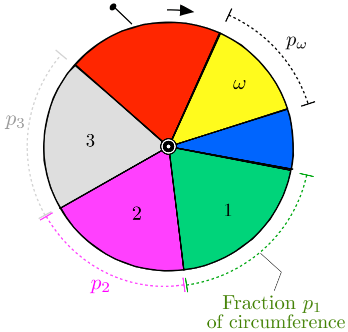

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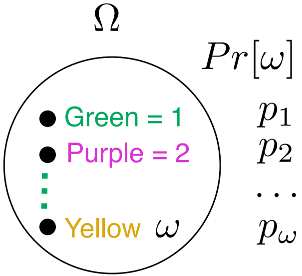

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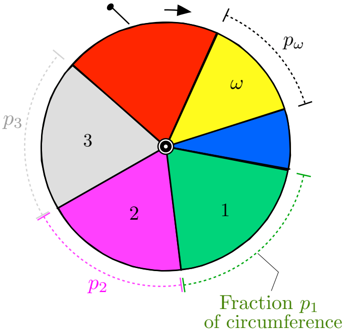

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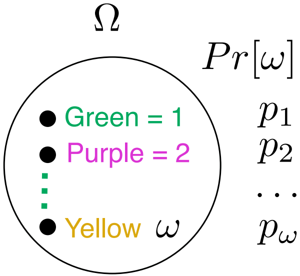

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Lecture 15: Summary

Modeling Uncertainty: Probability Space

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1. Random Experiment

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Modeling Uncertainty: Probability Space

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CS70: On to Calculation.

Events, Conditional Probability, Independence, Bayes' Rule

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Events, Conditional Probability, Independence, Bayes' Rule

1. Probability Basics Review
2. Events
3. Conditional Probability
4. Independence of Events
5. Bayes' Rule

Probability Basics Review

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Set notation review

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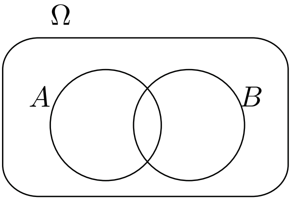
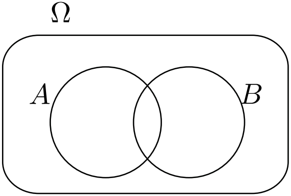


Figure: Two events

Set notation review



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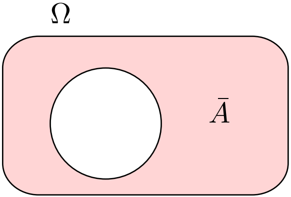
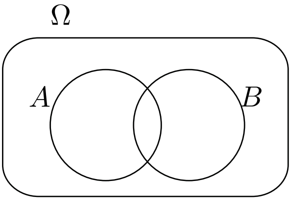


Figure: Complement (not)

Set notation review



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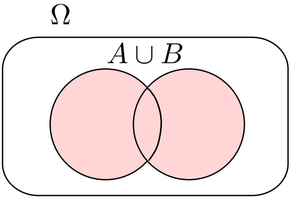
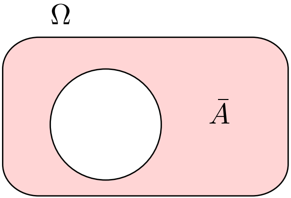


Figure: Union (or)



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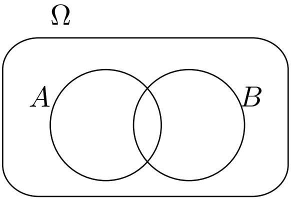

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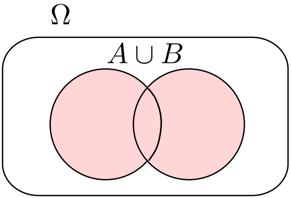

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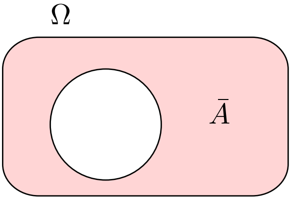

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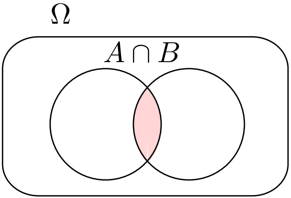
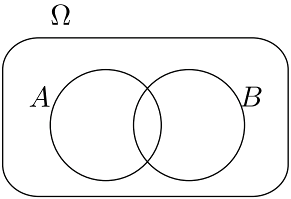


Figure: Intersection (and)

Set notation review



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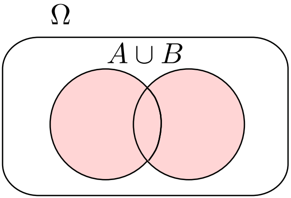

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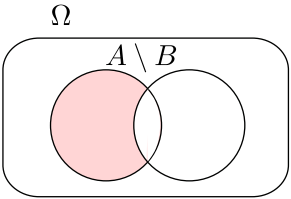


Figure: Difference (*A*, not *B*)

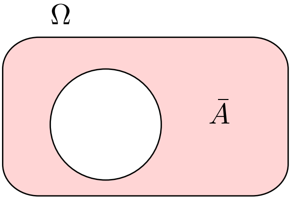


Figure: Complement (not)

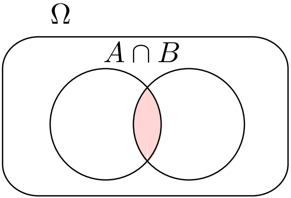


Figure: Intersection (and)

Set notation review

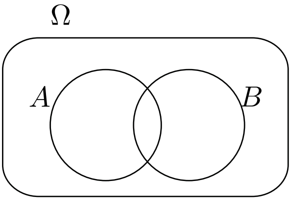


Figure: Two events

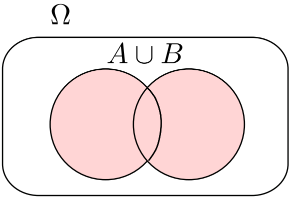


Figure: Union (or)

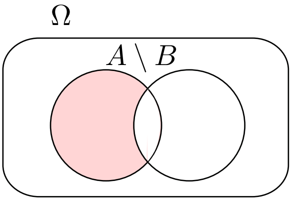
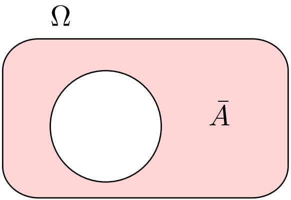


Figure: Difference (A, not B)



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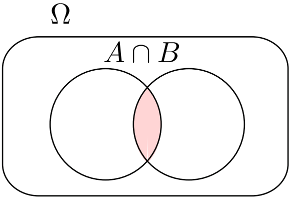

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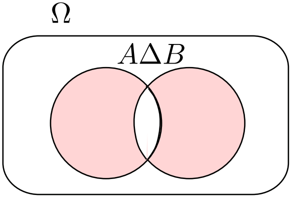

Figure: Symmetric difference (only one)

Probability of exactly one ‘heads’ in two coin flips?

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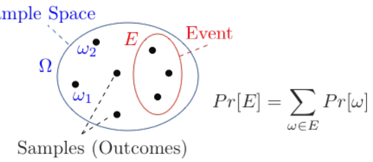

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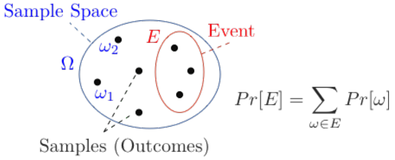

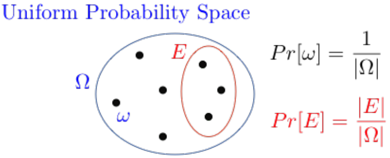

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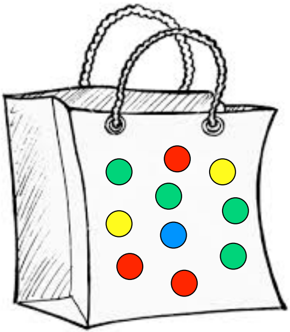
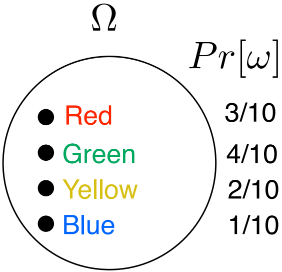


Physical experiment



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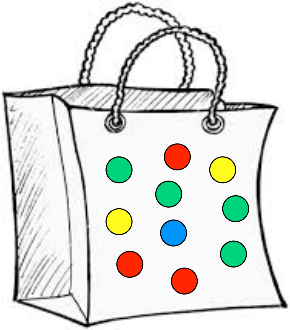

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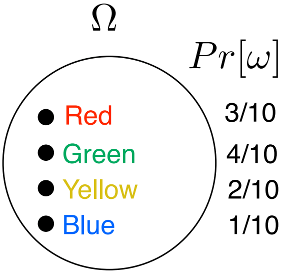

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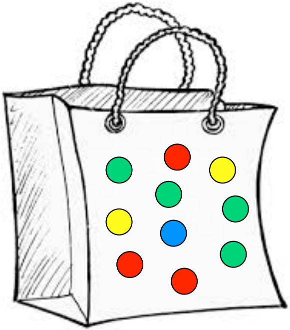

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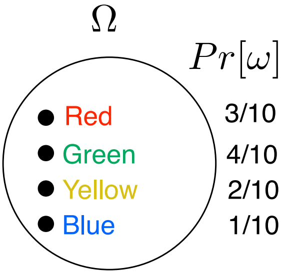

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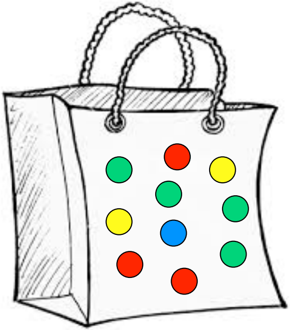

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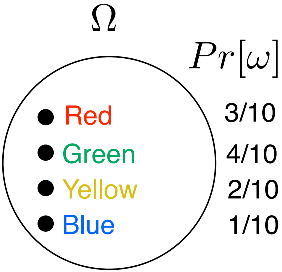

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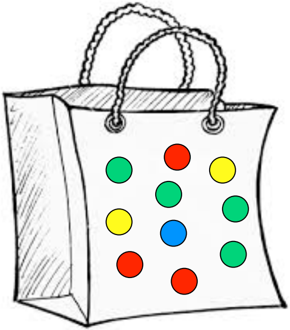

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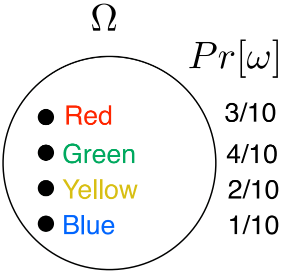

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$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] =$$

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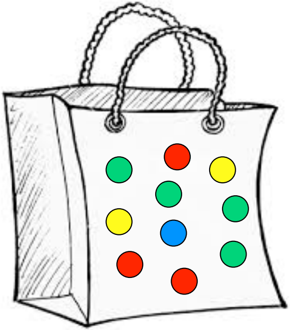

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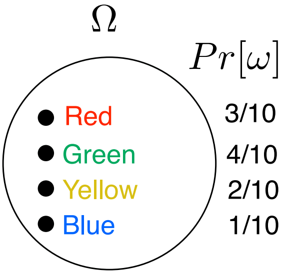

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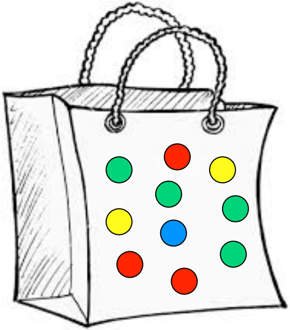

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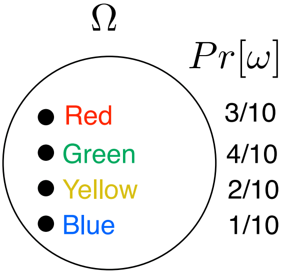

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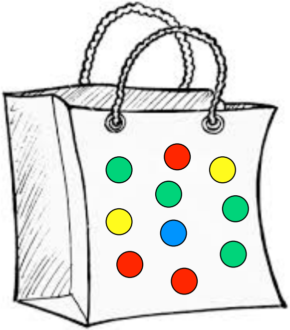

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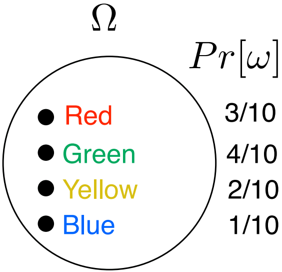

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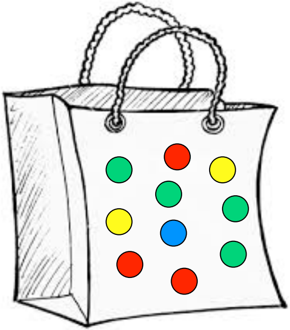

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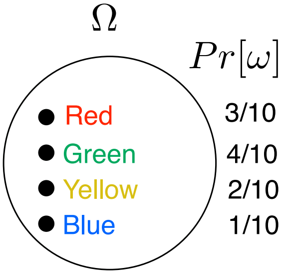

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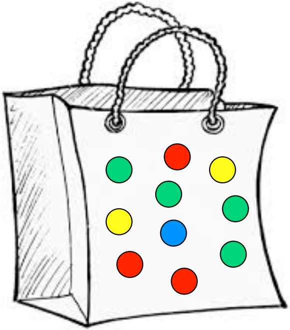

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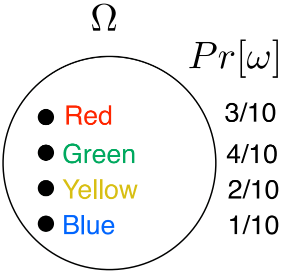

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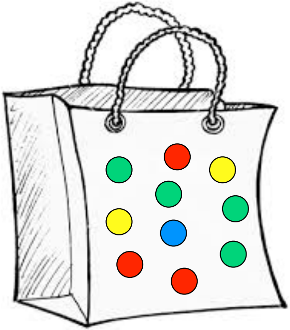

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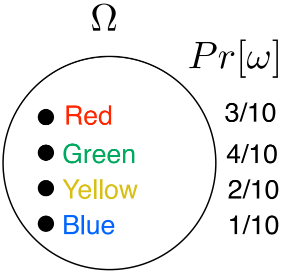

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Uniform probability space:
$Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$.

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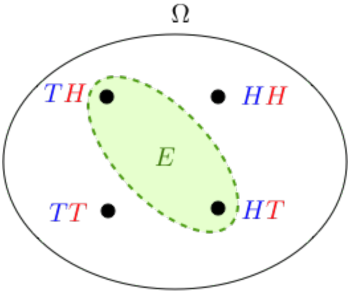

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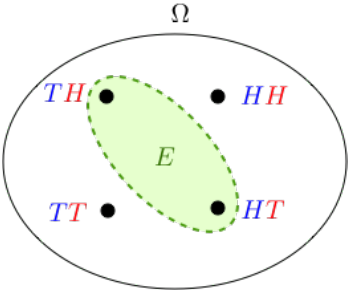

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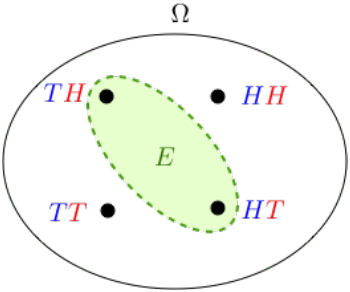

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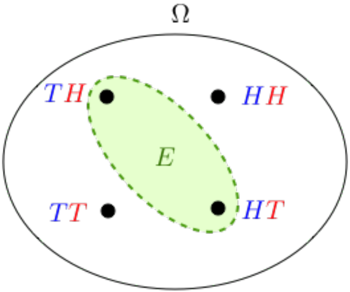

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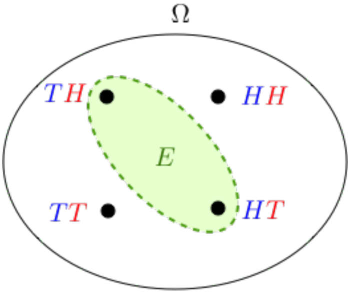

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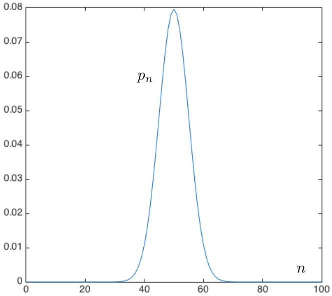

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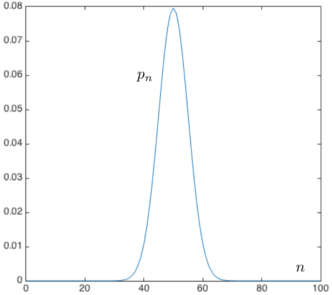

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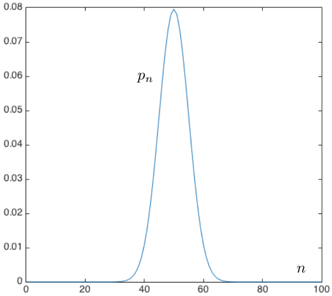

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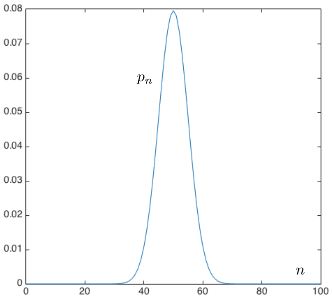

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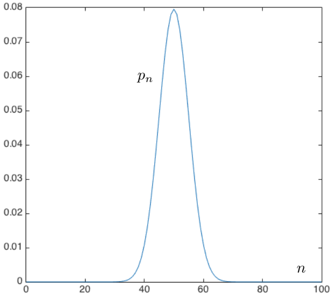

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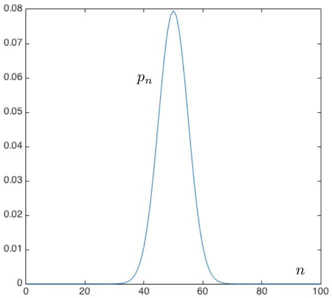

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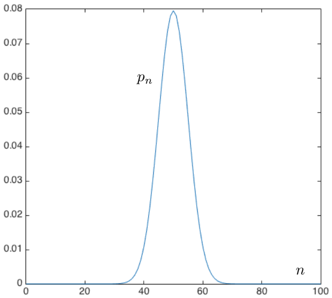

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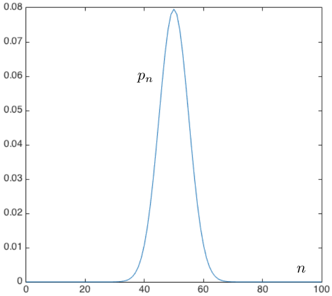

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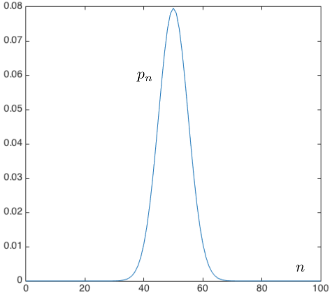

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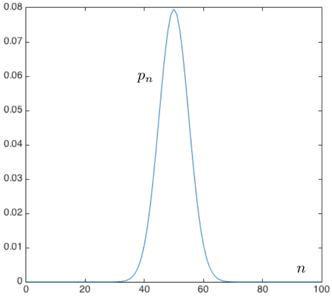

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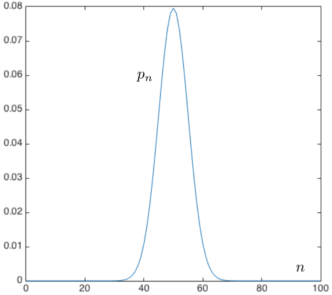

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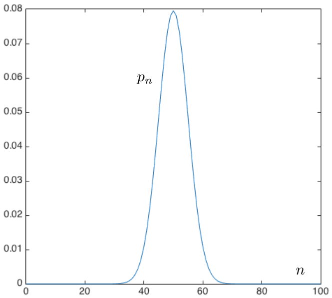

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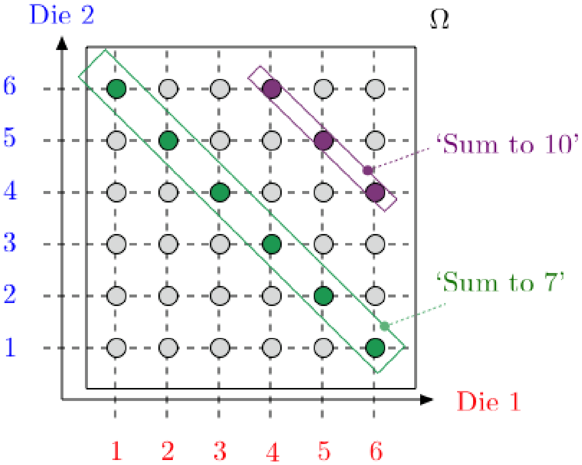

$Pr[\text{Sum to } 7] = \frac{6}{36}$ $Pr[\text{Sum to } 10] = \frac{3}{36}$

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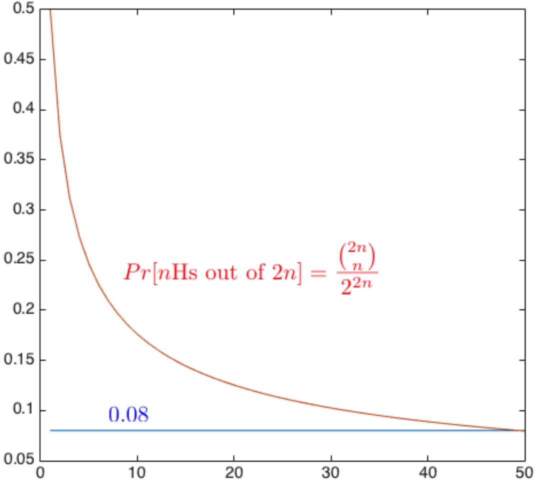

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(b) $Pr[A_1 \cup \cdots \cup A_n] \leq Pr[A_1] + \cdots + Pr[A_n]$;

(union bound)

(c) If $A_1, \ldots A_N$ are a partition of Ω, i.e.,
pairwise disjoint and $\cup_{m=1}^{N} A_m = \Omega$, then

$$Pr[B] = Pr[B \cap A_1] + \cdots + Pr[B \cap A_N].$$

(law of total probability)

Proof:

(b) is obvious.

Proofs for (a) and (c)?

Consequences of Additivity

Theorem

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Proofs for (a) and (c)? Next...

Inclusion/Exclusion

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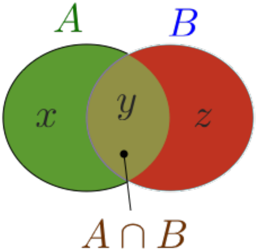

$$\begin{aligned} Pr[A] &= x + y \\ Pr[B] &= y + z \\ Pr[A \cap B] &= y \\ Pr[A \cup B] &= x + y + z \end{aligned}$$

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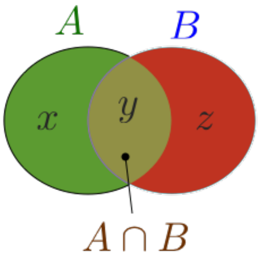

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Another view.

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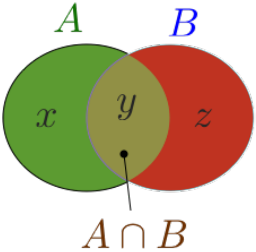

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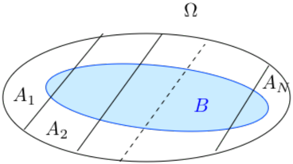
Another view. Any $\omega \in A \cup B$ is in $A \cap \overline{B}$, $A \cup B$, or $\overline{A} \cap B$. So, add it up.

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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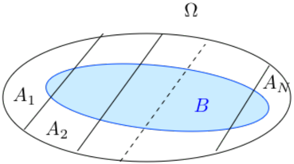

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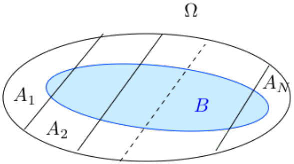

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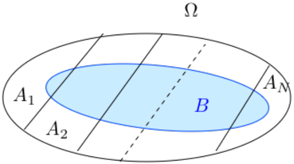

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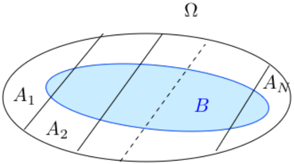

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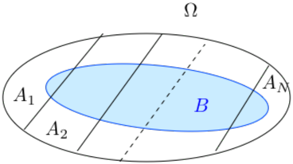

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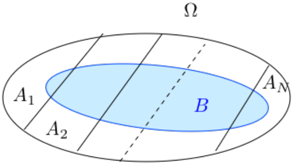

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Add it up.

Roll a Red and a Blue Die.

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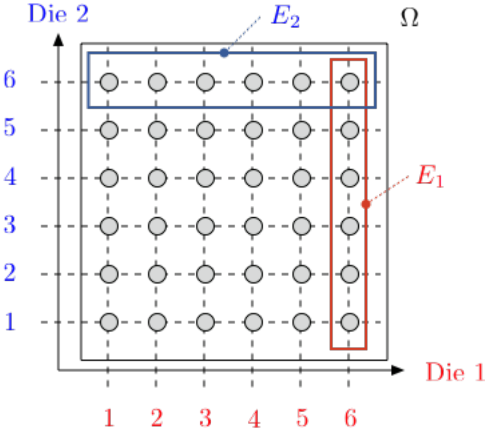

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

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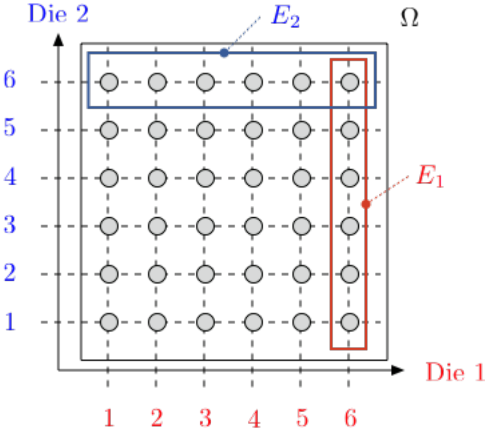

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$E_1 =$ 'Red die shows 6';

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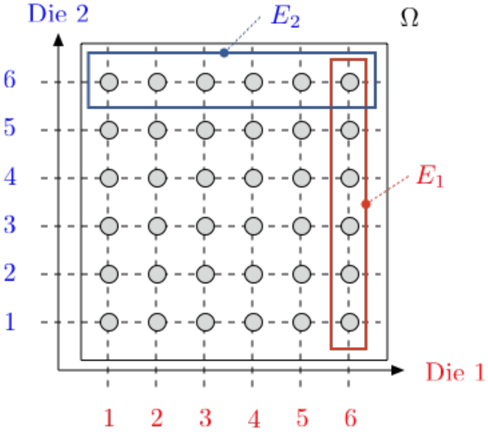

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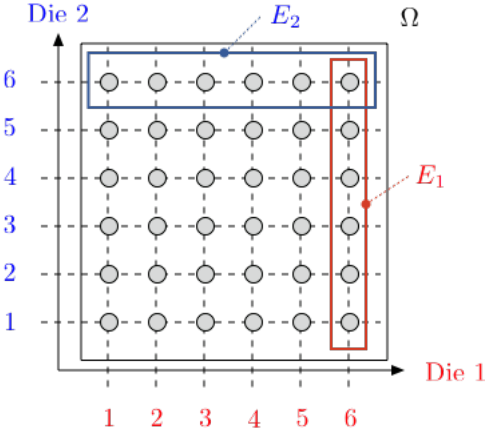

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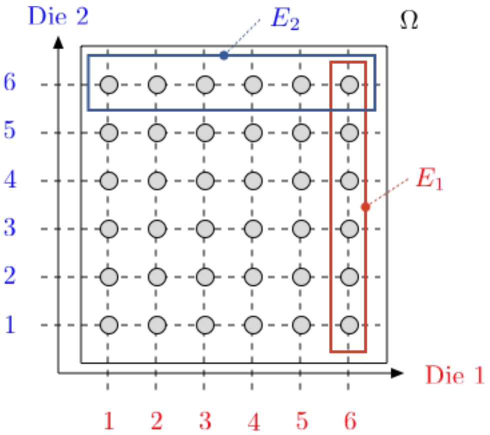

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$$Pr[E_1] = \frac{6}{36},$$

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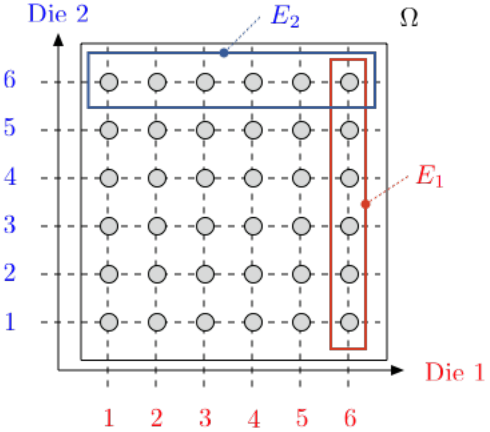

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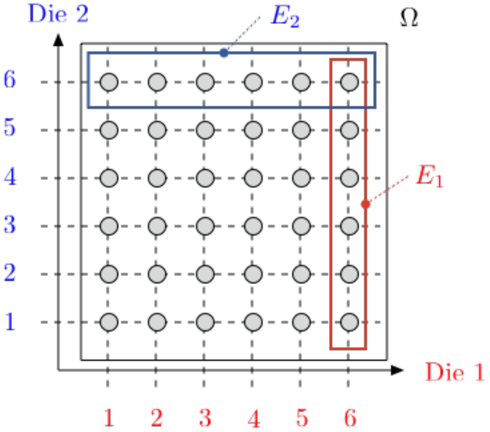

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$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Conditional probability: example.

Two coin flips.

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Two coin flips. First flip is heads.

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Two coin flips. First flip is heads. Probability of two heads?

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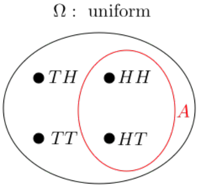

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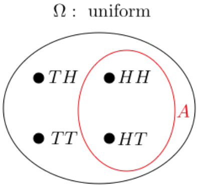

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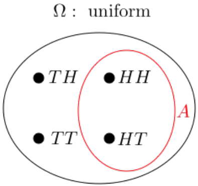

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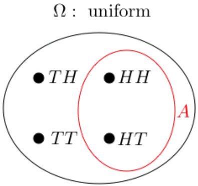

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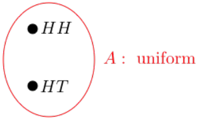

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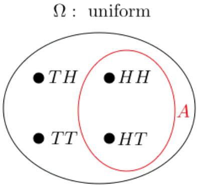

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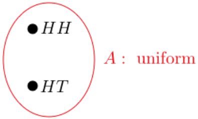

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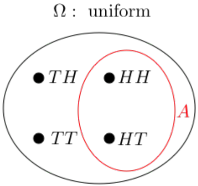

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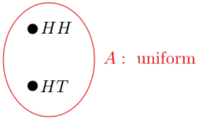

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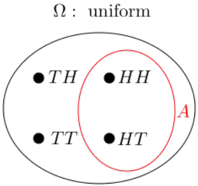

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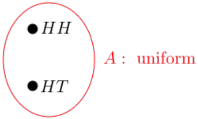

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The probability of B given A

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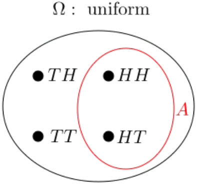

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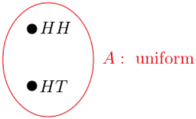

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The probability of two heads if the first flip is heads.
The probability of B given A is $1/2$.

A similar example.

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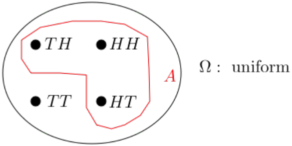

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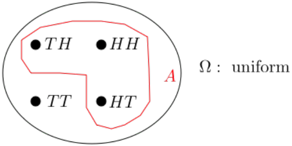

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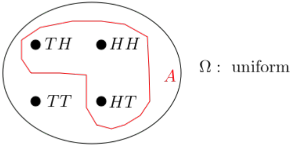

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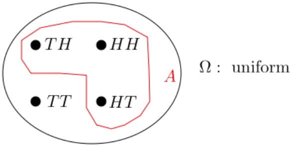

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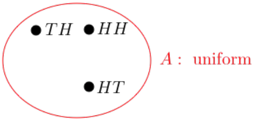

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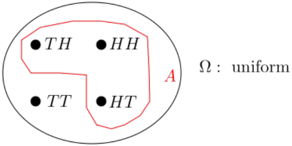

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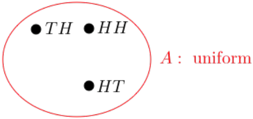

Event $B =$ two heads.

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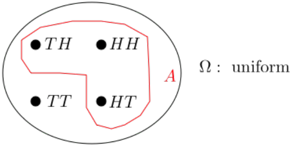

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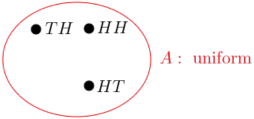

Event $B =$ two heads.

The probability of two heads if at least one flip is heads.

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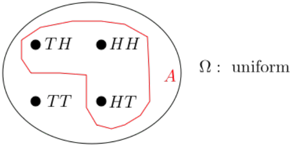
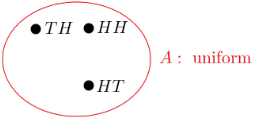


New sample space: A; uniform still.



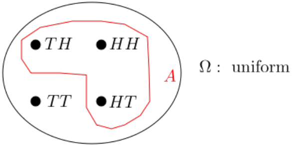
Event $B =$ two heads.

The probability of two heads if at least one flip is heads.
The probability of B given A

A similar example.

Two coin flips. At least one of the flips is heads.
$\rightarrow$ Probability of two heads?

$\Omega = \{HH, HT, TH, TT\}$; uniform.
Event $A =$ at least one flip is heads. $A = \{HH, HT, TH\}$.



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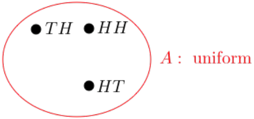

Event $B =$ two heads.

The probability of two heads if at least one flip is heads.
The probability of B **given** A is $1/3$.

Conditional Probability: A non-uniform example

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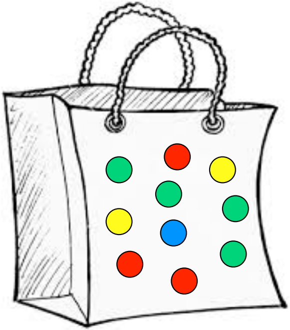
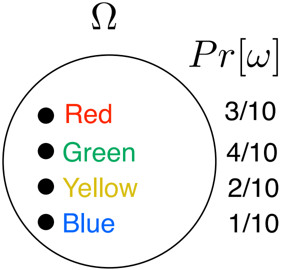
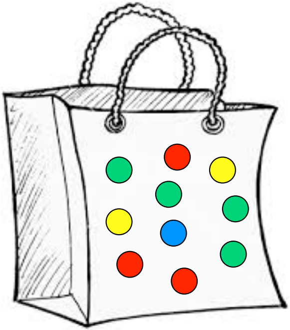


Physical experiment



Probability model

Conditional Probability: A non-uniform example



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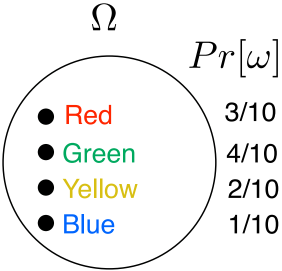

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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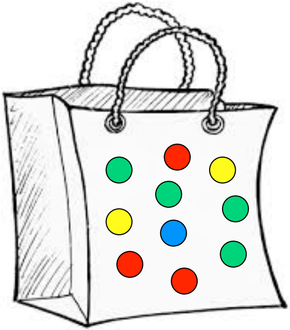

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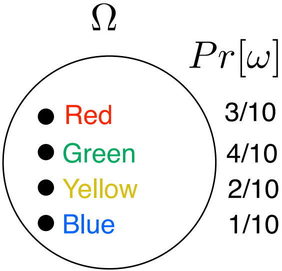
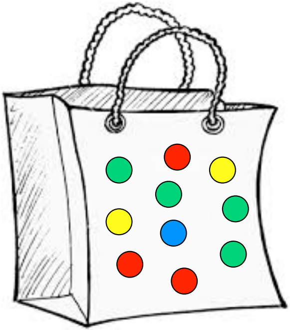


Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red} | \text{Red or Green}] =$$

Conditional Probability: A non-uniform example



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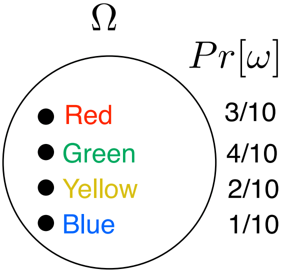
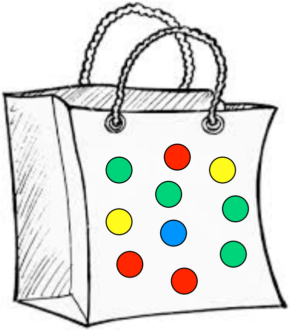


Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} =$$

Conditional Probability: A non-uniform example



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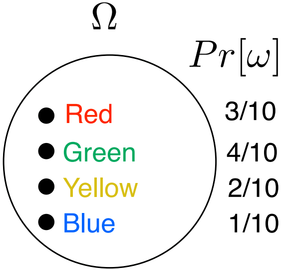

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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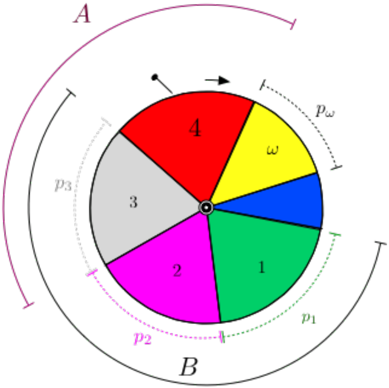

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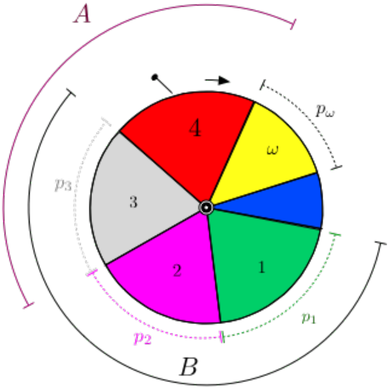

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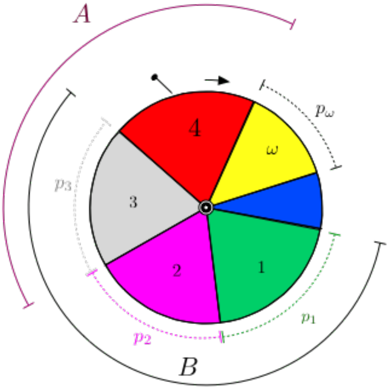

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

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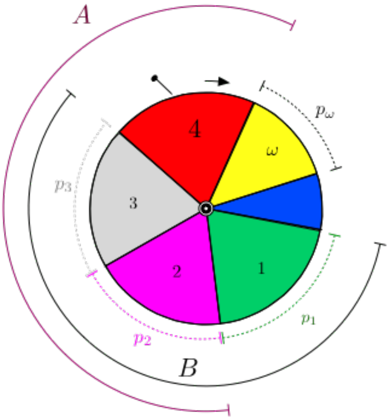

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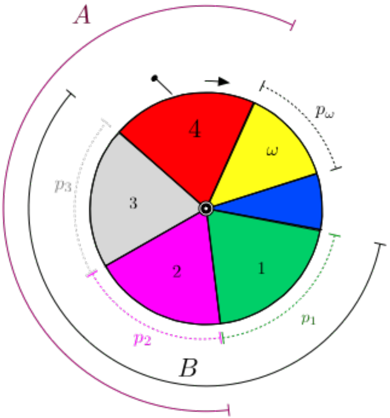

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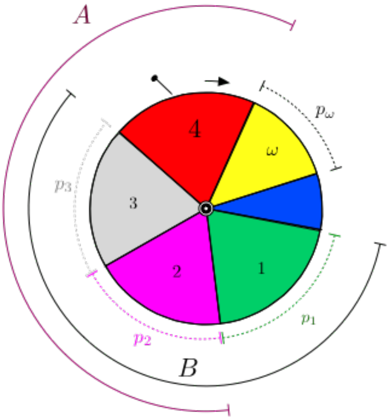

$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Conditional Probability.

Definition: The **conditional probability** of B given A is

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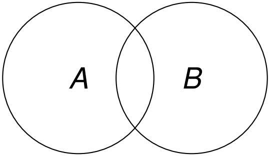

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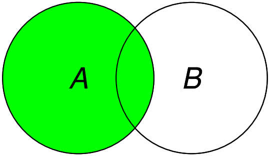

In A!

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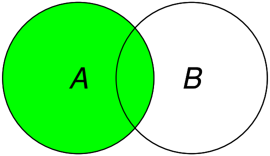

In A!
In B?

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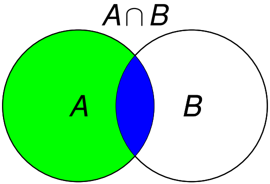

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Must be in $A \cap B$.

Conditional Probability.

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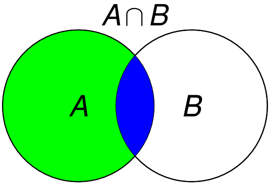
$$Pr[B|A] = \frac{Pr[A\cap B]}{Pr[A]}$$



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Must be in $A\cap B$.

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More fun with conditional probability.

Toss a red and a blue die, sum is 4,

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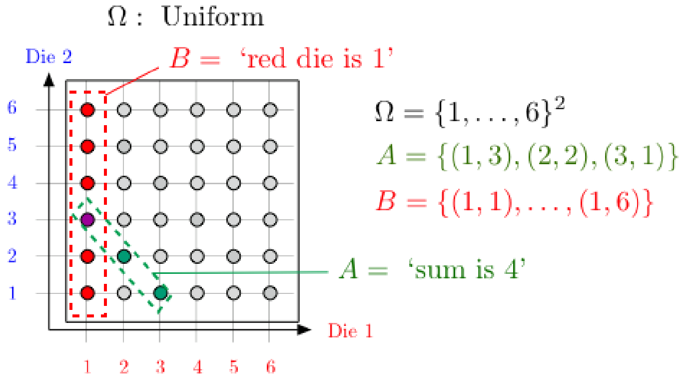

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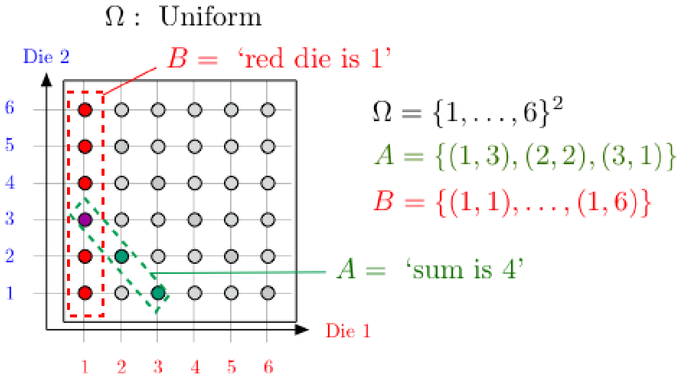

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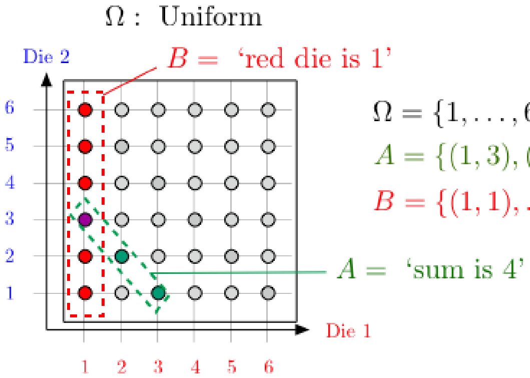

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1,3), (2,2), (3,1)\}$

$B = \{(1,1), \ldots, (1,6)\}$

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{3}$; versus $Pr[B] = 1/6$.

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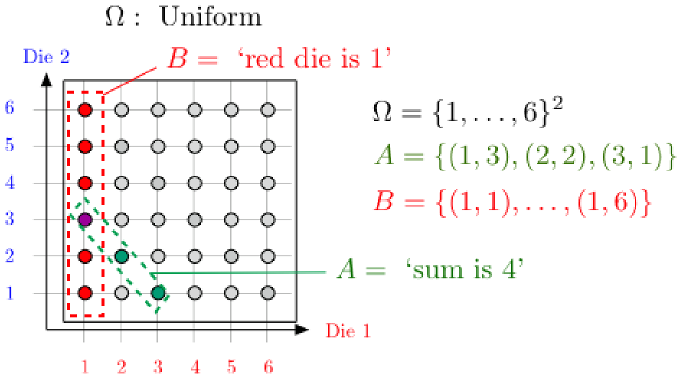

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{3}$; versus $Pr[B] = 1/6$.

B is more likely given *A*.

Yet more fun with conditional probability.

Toss a red and a blue die, sum is 7,
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Toss a red and a blue die, sum is 7,
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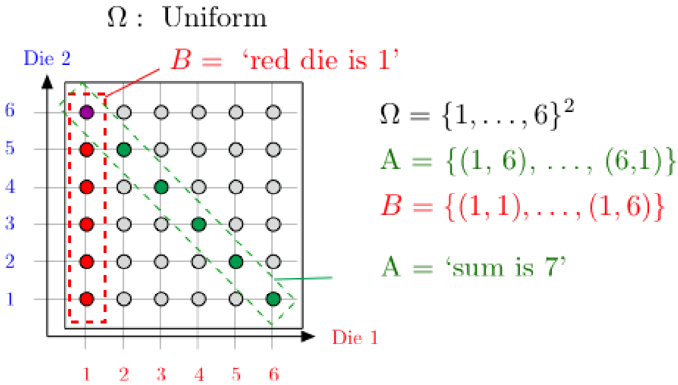

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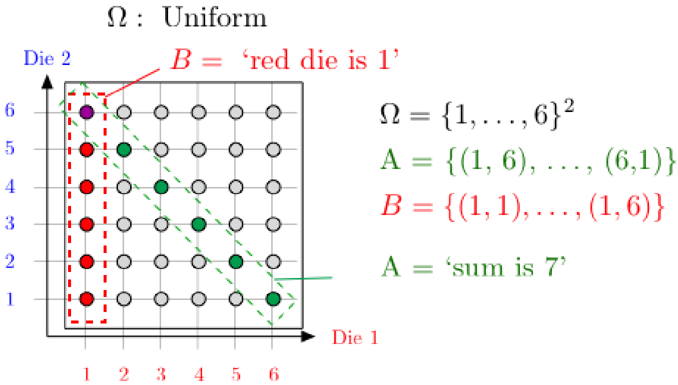

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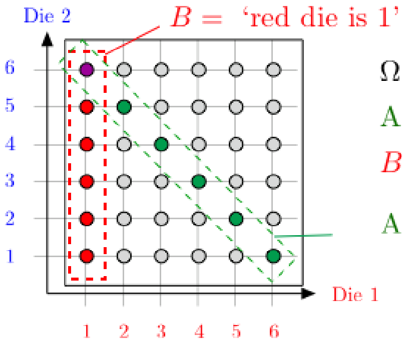
$\Omega:$ Uniform



$\Omega = \{1,\ldots,6\}^2$

A $= \{(1,6),\ldots,(6,1)\}$

$B = \{(1,1),\ldots,(1,6)\}$

A = 'sum is 7'

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{6}$; versus $Pr[B] = \frac{1}{6}$.

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Toss a red and a blue die, sum is 7,
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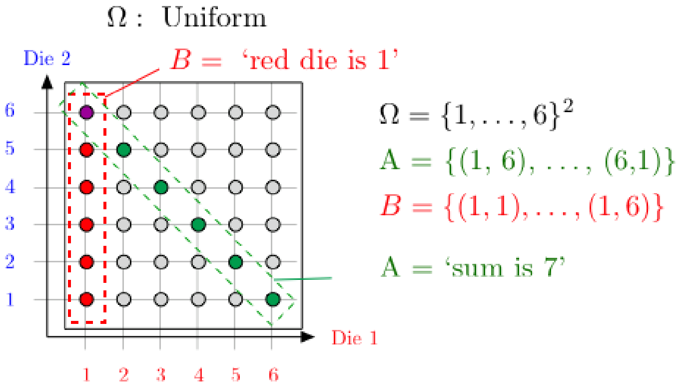

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{6}$; versus $Pr[B] = \frac{1}{6}$.

Observing *A* does not change your mind about the likelihood of *B*.

Emptiness..

Suppose I toss 3 balls into 3 bins.

Emptiness..

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$A =$"1st bin empty";

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Suppose I toss 3 balls into 3 bins.
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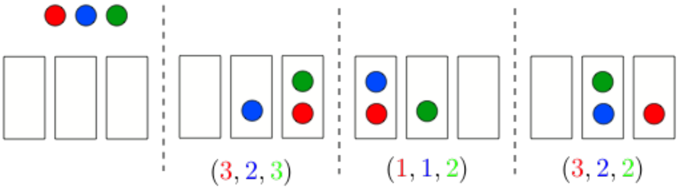
$$\Omega = \{1,2,3\}^3$$



$$\omega = (\text{bin of red ball}, \text{bin of blue ball}, \text{bin of green ball})$$

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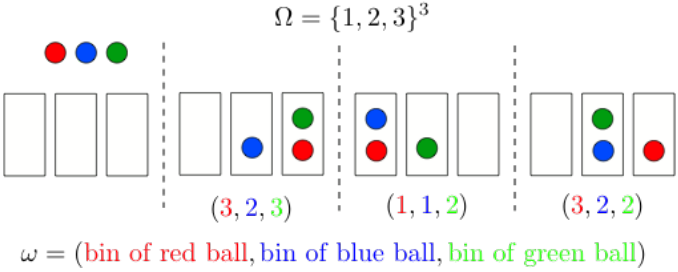

$Pr[B]$

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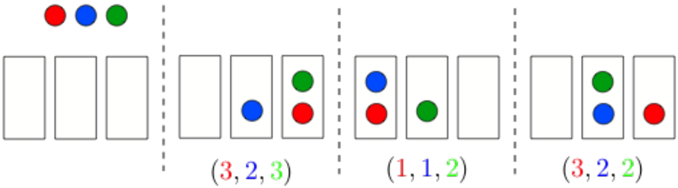

$$\omega = (\text{bin of red ball}, \text{bin of blue ball}, \text{bin of green ball})$$

$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] =$

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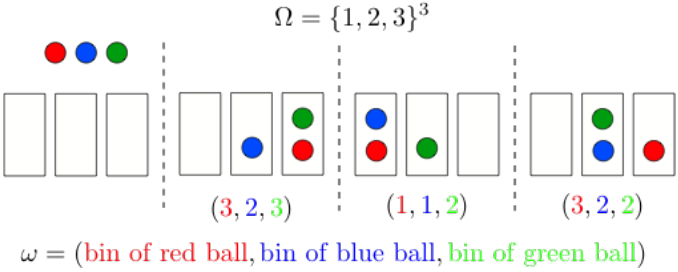

$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] = Pr[\{1,3\}^3] =$

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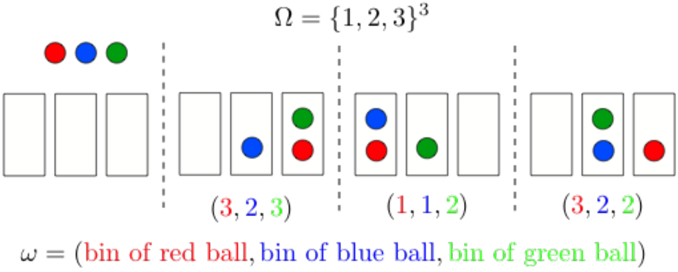

$$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] = Pr[\{1,3\}^3] = \frac{8}{27}$$

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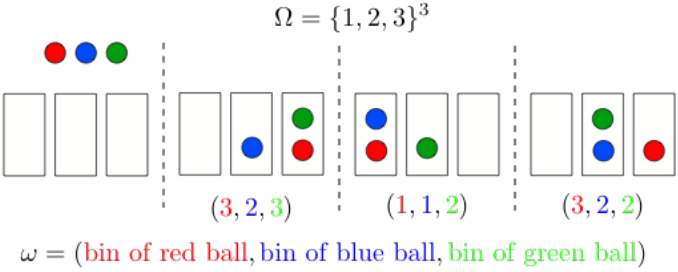

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$Pr[A \cap B]$

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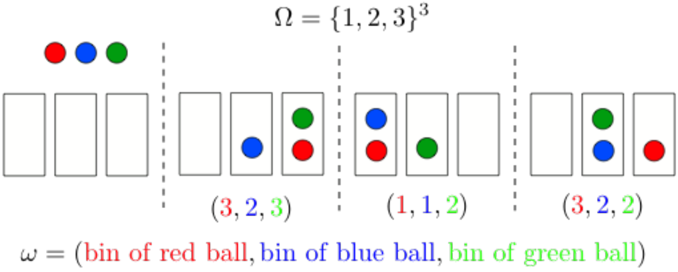

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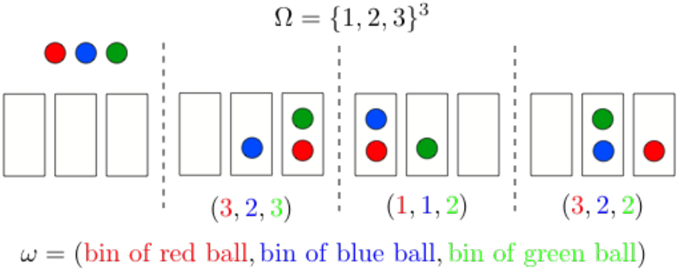

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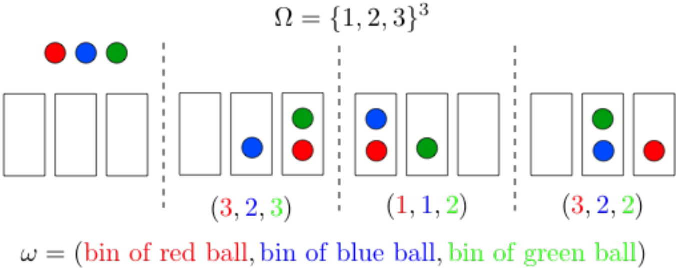

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$Pr[A|B]$

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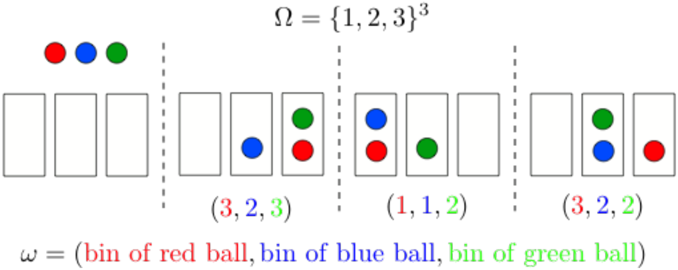

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$Pr[A \cap B] = Pr[(3,3,3)] = \frac{1}{27}$

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$

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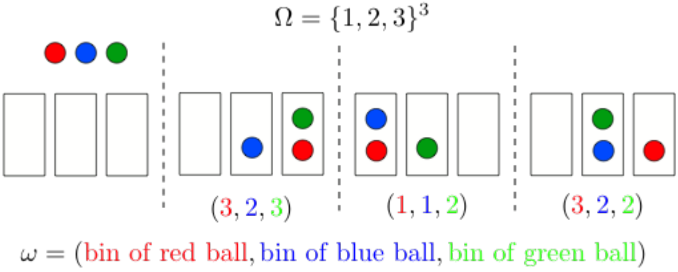

$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] = Pr[\{1,3\}^3] = \frac{8}{27}$

$Pr[A \cap B] = Pr[(3,3,3)] = \frac{1}{27}$

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{(1/27)}{(8/27)} = 1/8;$

Emptiness..

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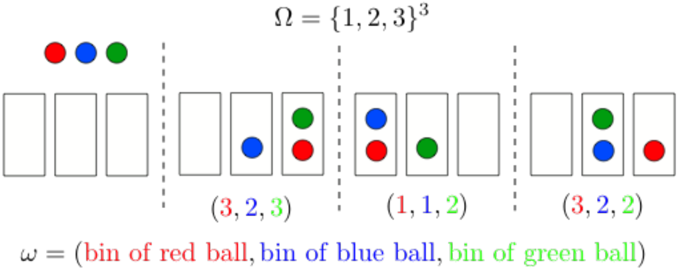

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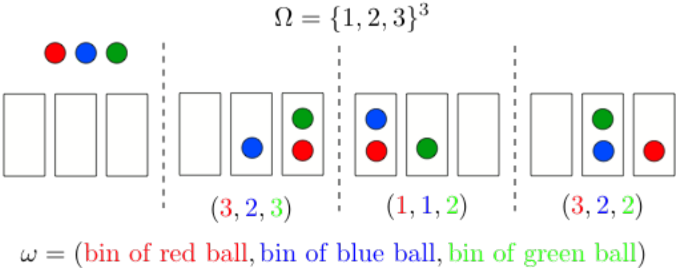

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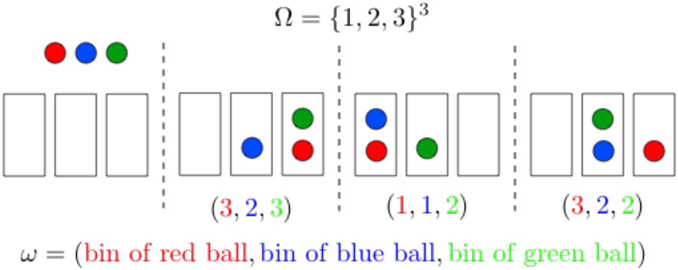

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A is less likely given B: If second bin is empty the first is more likely to have balls in it.

Three Card Problem

Three cards: Red/Red, Red/Black, Black/Black.
Pick one at random and place on the table. The upturned side is a Red. What is the probability that the other side is Black?
Can't be the BB card, so...prob should be 0.5, right?
R: upturned card is Red; RB: the Red/Black card was selected.
Want $P(RB|R)$.
What's wrong with the reasoning that leads to $\frac{1}{2}$?

$$\begin{aligned} P(RB|R) &= \frac{P(RB \cap R)}{P(R)} \\ &= \frac{\frac{1}{3}\frac{1}{2}}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}(0)} \\ &= \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \end{aligned}$$

Once you are given R: it is twice as likely that the RR card was picked.

Gambler's fallacy.

Flip a fair coin 51 times.

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The likelihood of 51st heads does not depend on the previous flips.

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so that the result holds for $n + 1$. $\square$

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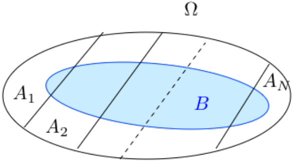
Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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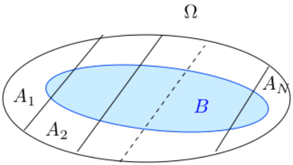

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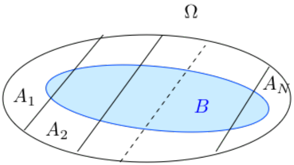

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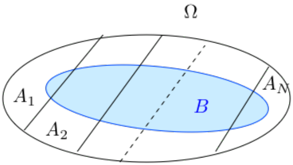

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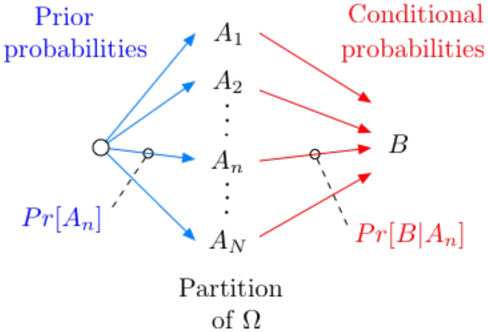

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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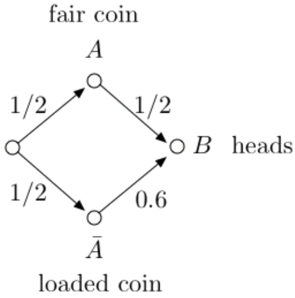

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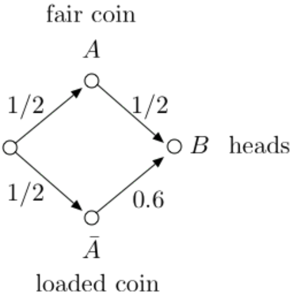

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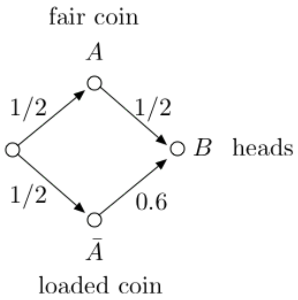

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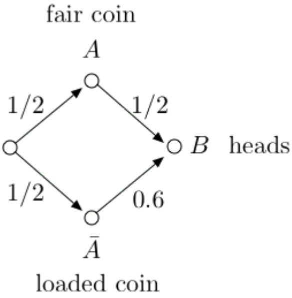

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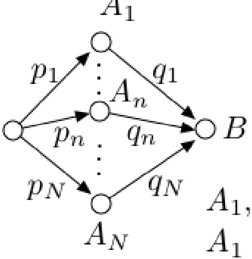

$$p_n = Pr[A_n]$$

$$q_n = Pr[B|A_n]$$

$A_1, \ldots, A_N$ disjoint

$A_1 \cup \cdots \cup A_N = \Omega$

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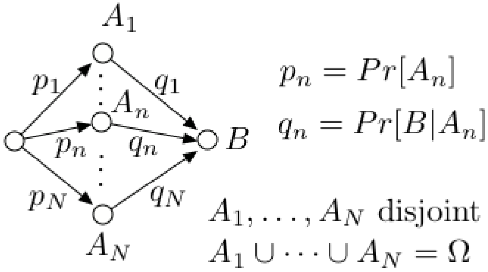

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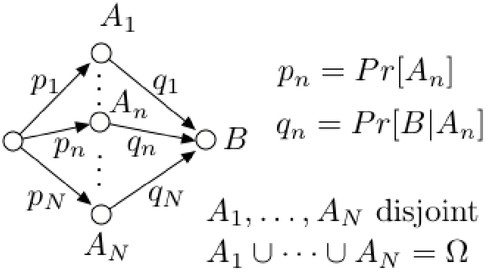

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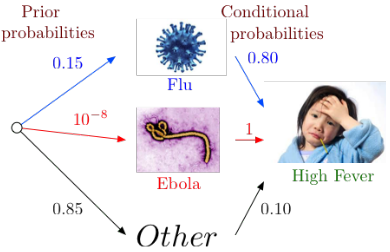
Why do you have a fever?



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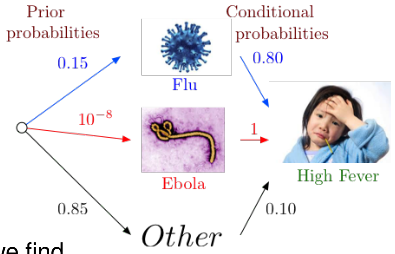

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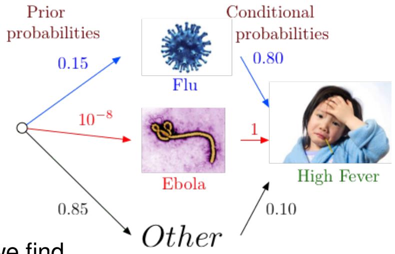

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$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

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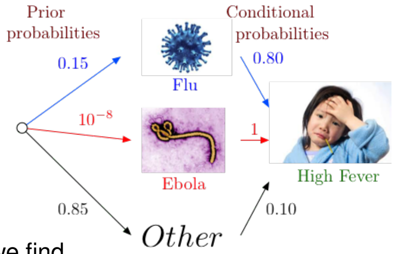

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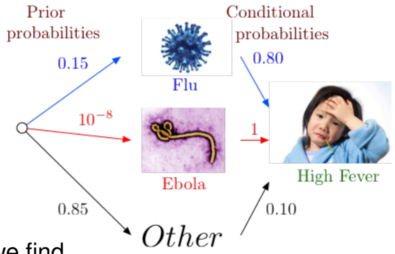

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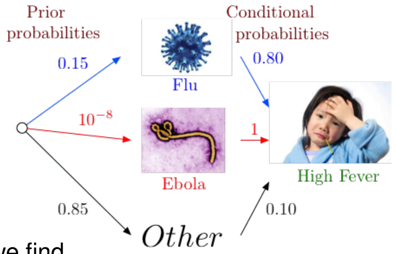

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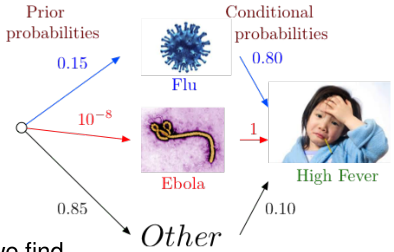

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Bayes’ Rule Operations

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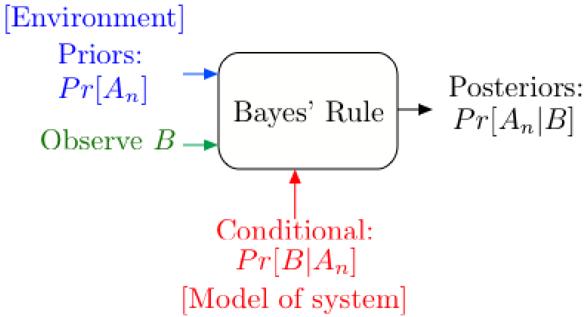

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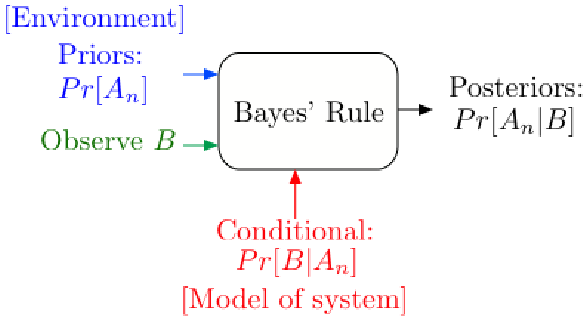

Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$\xrightarrow{Pr[B|A]}$

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Let's watch TV!!

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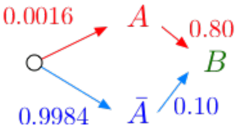

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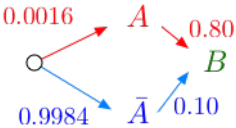

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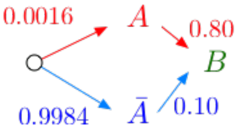

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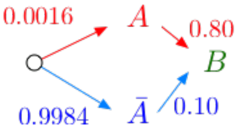

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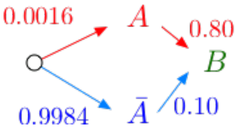

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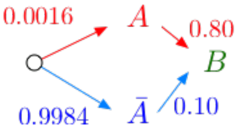

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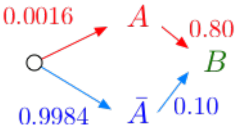

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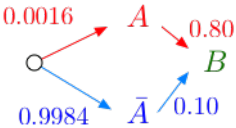

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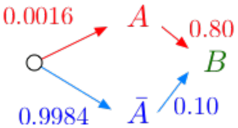

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Events, Conditional Probability, Independence, Bayes' Rule

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Key Ideas:

- Conditional Probability:

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- All these are possible:

  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A]$.