Continuing Probability.

Wrap up: Total Probability and Conditional Probability.
Product Rule, Correlation, Independence, Bayes’ Rule.

Total probability
Assume that \( \Omega \) is the union of the disjoint sets \( A_1, \ldots, A_N \).
Then,
\[
\Pr[B] = \Pr[A_1 \cap B] + \cdots + \Pr[A_N \cap B].
\]
Indeed, \( B \) is the union of the disjoint sets \( A_n \cap B \) for \( n = 1, \ldots, N \).

Conditional probability: example.
Two coin flips. First flip is heads. Probability of two heads?
\( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space.
Event \( A \) = first flip is heads: \( A = \{HH, HT\} \).
New sample space: \( A \); uniform still.
Event \( B \) = two heads.
The probability of two heads if the first flip is heads.
The probability of \( B \) given \( A \) is \( \frac{1}{2} \).

Emptiness..
Suppose I toss 3 balls into 3 bins.
\( A \) = “1st bin empty”; \( B \) = “2nd bin empty.”
What is \( \Pr[A | B] \)?
\[
\Pr[B] = \Pr\left[ \left\{ (a, b, c) \mid a, b, c \in \{1, 3\} \right\} \right] = \frac{8}{27}
\]
\[
\Pr[A \cap B] = \Pr\left[ \left\{ (3, 3, 3) \right\} \right] = \frac{1}{27}
\]
\[
\Pr[A | B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{1}{3} \; vs. \; \Pr[A] = \frac{8}{27}.
\]
A is less likely given \( B \): If second bin is empty the first is more likely to have balls in it.

Conditional Probability.
Definition: The conditional probability of \( B \) given \( A \) is
\[
\Pr[B | A] = \frac{\Pr[A \cap B]}{\Pr[A]}
\]

Three Card Problem
Three cards: Red/Red, Red/Black, Black/Black.
Pick one at random and place on the table. The upturned side is a Red. What is the probability that the other side is Black?
Can’t be the BB card, so... prob should be 0.5, right?
\( R \) = upturned card is Red; \( RB \) = the Red/Black card was selected.
Want \( P(RB | R) \).
What’s wrong with the reasoning that leads to \( \frac{3}{4} \)?
\[
P(RB | R) = \frac{P(RB \cap R)}{P(R)} = \frac{1}{1} = \frac{1}{2} \; vs. \; \frac{3}{4}
\]
Once you are given \( R \): it is twice as likely that the RR card was picked.
Gambler’s fallacy.

Flip a fair coin 51 times. A = “first 50 flips are heads”
B = “the 51st is heads”
Pr(B|A) ?
A = {HH ··· HT, HH ··· HH}
B ∩ A = {HH ··· HH}
Uniform probability space.
Pr(B|A) = \frac{|B \cap A|}{|A|} = \frac{1}{2}.
Same as Pr[B].
The likelihood of 51st heads does not depend on the previous flips.

Product Rule

Recall the definition:
Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.
Hence,
Pr[A \cap B] = Pr[A]Pr[B|A].
Consequently,
Pr[A \cap B \cap C] = Pr[(A \cap B) \cap C]
= Pr[A \cap B]Pr[C|A \cap B]
= Pr[A]Pr[B|A]Pr[C|A \cap B].

Correlation

An example.
Random experiment: Pick a person at random.
Event A: the person has lung cancer.
Event B: the person is a heavy smoker.
Fact: 
Pr[A ∩ B] = 1.17 × Pr[A].
Conclusion:
- Smoking increases the probability of lung cancer by 17%.
- Smoking causes lung cancer.

Causality vs. Correlation

Events A and B are positively correlated if
Pr[A ∩ B] > Pr[A]Pr[B].
(E.g., smoking and lung cancer.)
A and B being positively correlated does not mean that A causes B or that B causes A.

Other examples:
- Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?
Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.

Then,
$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. Thus,

Is your coin loaded?

A picture:

Imagine 100 situations, among which $m := 100(1/2)(1/2)$ are such that $A$ and $B$ occur and $n := 100(1/2)(0.6)$ are such that $A$ and $B$ occur. Hence,
$$Pr[A|B] = \frac{m}{m+n} = \frac{(1/2)(1/2)}{(1/2)(1/2)+(1/2)(0.6)}.$$

Independence

Definition: Two events $A$ and $B$ are independent if $Pr[A \cap B] = Pr[A]Pr[B].$

Examples:
- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent;
- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent;
- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent;
- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are not independent;

Independence and conditional probability

Fact: Two events $A$ and $B$ are independent if and only if $Pr[A|B] = Pr[A]$.

Indeed:
$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{Pr[A]}{Pr[B]} \iff Pr[A \cap B] = Pr[A]Pr[B].$$

Is your coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads. What is the probability that it is fair?

Analysis:
- $A =$ ‘coin is fair’, $B =$ ‘outcome is heads’
- We want to calculate $P[A|B]$
- We know $P[B|A] = 1/2$, $P[B|\overline{A}] = 0.6$, thus $P[A] = 1/2 = P[\overline{A}]$
- Now,
$$Pr[B] = Pr[A \cap B] + Pr[\overline{A} \cap B] = Pr[A]Pr[B|A] + Pr[\overline{A}]Pr[B|\overline{A}] = (1/2)(1/2) + (1/2)(0.6) = 0.55.$$
- Thus,
$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2)+(1/2)(0.6)} \approx 0.45.$$
Bayes Rule

Another picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$.

Imagine 100 situations, among which 100 $p_n q_n$ are such that $A_n$ and $B$ occur, for $n = 1, \ldots, N$.

Thus, among the $\sum_m p_m q_m$ situations where $B$ occurred, there are $100 p_n q_n$ where $A_n$ occurred.

Hence,

$$\Pr[A_n | B] = \frac{p_n q_n}{\sum m p_m q_m}.$$

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Bayes’ Rule Operations

Bayes’ Rule is the canonical example of how information changes our opinions.

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Thomas Bayes

A Bayesian picture of Thomas Bayes.


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Why do you have a fever?

Using Bayes’ rule, we find

$$Pr[\text{Flu} | \text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 0.10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

$$Pr[\text{Ebola} | \text{High Fever}] = \frac{0.15 \times 0.80 + 0.10^{-8} \times 1 + 0.85 \times 0.1}{0.15 \times 0.80 + 0.10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}$$

$$Pr[\text{Other} | \text{High Fever}] = \frac{0.15 \times 0.80 \times 10^{-8} \times 1 + 0.85 \times 0.1}{0.15 \times 0.80 \times 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42$$

These are the posterior probabilities. One says that ‘Flu’ is the Most Likely a Posterior (MAP) cause of the high fever.

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Testing for disease.

Let’s watch TV!!
Random Experiment: Pick a random male.
Outcomes: (test, disease)

- $A$ - prostate cancer.
- $B$ - positive PSA test.

- $Pr[A] = 0.0016$, (.16 \% of the male population is affected.)
- $Pr[B | A] = 0.80$ (80\% chance of positive test with disease.)
- $Pr[B | \overline{A}] = 0.10$ (10\% chance of positive test without disease.)


Positive PSA test ($B$). Do I have disease?

$$Pr[A | B] ???$$
Using Bayes’ rule, we find
\[ P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = 0.013. \]
A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?
Impotence...
Incontinence..
Death.

Summary
Events, Conditional Probability, Independence, Bayes’ Rule
Key Ideas:
▶ Conditional Probability:
\[ Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} \]
▶ Independence: \[ Pr[A \cap B] = Pr[A]Pr[B]. \]
▶ Bayes’ Rule:
\[ Pr[A_i|B] = \frac{Pr[A_i]Pr[B|A_i]}{\sum_i Pr[A_i]Pr[B|A_i]} \]
\[ Pr[A_i|B] = \text{posterior probability}; \ Pr[A_i] = \text{prior probability} . \]
▶ All these are possible:
\[ Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A]. \]