Continuing Probability.

Wrap up: Total Probability and Conditional Probability.
Continuing Probability.

Wrap up: Total Probability and Conditional Probability.

Product Rule, Correlation, Independence, Bayes’ Rule,
Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$. 
Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.

Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$
Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$.

Then, $Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B]$.

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. 
Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$. Then,

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\Pr[B] = \Pr[A_1 \cap B] + \cdots + \Pr[A_N \cap B].
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Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. 
Conditional probability: example.

Two coin flips.
Conditional probability: example.

Two coin flips. First flip is heads.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?
Conditional probability: example.

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\[ \Omega = \{HH, HT, TH, TT\}; \]
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?

\[ \Omega = \{HH, HT, TH, TT\} \]; Uniform probability space.
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Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A = \) first flip is heads:
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Two coin flips. First flip is heads. Probability of two heads?
\( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space.
Event \( A = \) first flip is heads: \( A = \{HH, HT\} \).

\[ \Omega: \text{uniform} \]

\[ \bullet TH \quad \bullet HH \]

\[ \bullet TT \quad \bullet HT \]

New sample space: \( A \);
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A = \) first flip is heads: \( A = \{HH, HT\} \).

New sample space: \( A \); uniform still.
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Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A = \) first flip is heads: \( A = \{HH, HT\} \).

\[
\begin{array}{c}
\Omega : \text{uniform} \\
\bullet TH & \bullet HH \\
\bullet TT & \bullet HT \\
\end{array}
\]

New sample space: \( A \); uniform still.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? $$\Omega = \{HH, HT, TH, TT\};$$ Uniform probability space. Event $$A =$$ first flip is heads: $$A = \{HH, HT\}.$$

New sample space: $$A;$$ uniform still.

Event $$B =$$ two heads.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? $\Omega = \{HH, HT, TH, TT\}$; Uniform probability space. Event $A =$ first flip is heads: $A = \{HH, HT\}$.

New sample space: $A$; uniform still.

Event $B =$ two heads. The probability of two heads if the first flip is heads.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A = \) first flip is heads: \( A = \{HH, HT\} \).

\[ \Omega : \text{uniform} \]

\[ \bullet TH \quad \bullet HH \]
\[ \bullet TT \quad \bullet HT \]

New sample space: \( A \); uniform still.

\[ \bullet HH \quad \bullet HT \]

Event \( B = \) two heads.

The probability of two heads if the first flip is heads. \textbf{The probability of } B \textbf{ given } A
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A = \) first flip is heads: \( A = \{HH, HT\} \).

\[ \Omega : \text{uniform} \]

\[ \begin{array}{c}
\bullet TH \\
\bullet TT
\end{array} \quad \begin{array}{c}
\bullet HH \\
\bullet HT
\end{array} \quad A \]

New sample space: \( A \); uniform still.

\[ \begin{array}{c}
\bullet HH \\
\bullet HT
\end{array} \quad \begin{array}{c}
A : \text{uniform}
\end{array} \]

Event \( B = \) two heads.

The probability of two heads if the first flip is heads. \( \textbf{The probability of } B \textbf{ given } A \) is \( 1/2 \).
Definition: The conditional probability of $B$ given $A$ is

\[ Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} \]
Emptiness.

Suppose I toss 3 balls into 3 bins.
Emptiness..

Suppose I toss 3 balls into 3 bins.

$A = \text{“1st bin empty”}$;
Emptiness.

Suppose I toss 3 balls into 3 bins. 
$A =$“1st bin empty”; $B =$“2nd bin empty.”
Suppose I toss 3 balls into 3 bins. 
\(A = \text{"1st bin empty"}; \ B = \text{"2nd bin empty."} \) What is \( Pr[A \mid B] \)?
Emptiness..

Suppose I toss 3 balls into 3 bins. 
$A =$“1st bin empty”; $B =$“2nd bin empty.” What is $Pr[A|B]$?

$\Omega = \{1, 2, 3\}^3$

$\omega = (\text{bin of red ball, bin of blue ball, bin of green ball})$
Emptiness..

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$Pr[B]$
Suppose I toss 3 balls into 3 bins. $A = \text{“1st bin empty”;}$ $B = \text{“2nd bin empty.”}$ What is $Pr[A | B]$?

$Pr[B] = Pr[\{(a, b, c) \mid a, b, c \in \{1, 3\}\}] = \frac{8}{27}$
Suppose I toss 3 balls into 3 bins. 

$A =$“1st bin empty”; $B =$“2nd bin empty.” What is $Pr[A|B]$?

$\Omega = \{1, 2, 3\}^3$

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$Pr[B] = Pr[\{(a, b, c) \mid a, b, c \in \{1, 3\}\}] = Pr[\{1, 3\}^3] = \frac{8}{27}$

$Pr[A \cap B] = \frac{1}{8}$; $Pr[A] = \frac{8}{27}$. $A$ is less likely given $B$: If second bin is empty the first is more likely to have balls in it.
Emptiness..

Suppose I toss 3 balls into 3 bins. 
\( A = \text{“1st bin empty”}; \; B = \text{“2nd bin empty.”} \) What is \( Pr[A|B] \)?

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\Omega = \{1, 2, 3\}^3
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\[
Pr[B] = Pr[\{(a, b, c) \mid a, b, c \in \{1, 3\}\}] = Pr[\{1, 3\}^3] = \frac{8}{27}
\]

\[
Pr[A \cap B] = Pr[(3, 3, 3)] = \quad
\]
Suppose I toss 3 balls into 3 bins. 

\( A = \text{“1st bin empty”}; \) \( B = \text{“2nd bin empty.”} \) What is \( Pr[A|B] \)?

\[
\begin{align*}
\Omega &= \{1, 2, 3\}^3 \\
\omega &= (\text{bin of red ball, bin of blue ball, bin of green ball}) \\
Pr[B] &= Pr[\{(a, b, c) \mid a, b, c \in \{1, 3\}\}] = Pr[\{1, 3\}^3] = \frac{8}{27} \\
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\end{align*}
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A = “1st bin empty”; B = “2nd bin empty.” What is $Pr[A|B]$?

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$Pr[A \cap B] = Pr[(3, 3, 3)] = \frac{1}{27}$

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{\frac{1}{27}}{\frac{8}{27}} = \frac{1}{8}$
Emptiness..

Suppose I toss 3 balls into 3 bins. 
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Pr[A \cap B] = Pr[(3, 3, 3)] = \frac{1}{27}
\]

\[
Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{1}{8}
\]

\( A \) is less likely given \( B \): If second bin is empty the first is more likely to have balls in it.
Suppose I toss 3 balls into 3 bins. 
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$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{(1/27)}{(8/27)} = 1/8$;
Emptiness..

Suppose I toss 3 balls into 3 bins. $A = “1st bin empty”; B = “2nd bin empty.”$ What is $Pr[A|B]?$

$\Omega = \{1, 2, 3\}^3$

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$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{(1/27)}{(8/27)} = 1/8; \text{ vs. } Pr[A] = \frac{8}{27}.$
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\[ \Pr[B] = \Pr[(a, b, c) \mid a, b, c \in \{1, 3\}] = \Pr[\{1, 3\}^3] = \frac{8}{27} \]

\[ \Pr[A \cap B] = \Pr[(3, 3, 3)] = \frac{1}{27} \]

\[ \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{(1/27)}{(8/27)} = 1/8; \text{ vs. } \Pr[A] = \frac{8}{27}. \]

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Suppose I toss 3 balls into 3 bins.  
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Pr[A \cap B] = Pr[(3, 3, 3)] = \frac{1}{27}
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\[
Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{\left(\frac{1}{27}\right)}{\left(\frac{8}{27}\right)} = 1/8; \ \text{vs.} \ Pr[A] = \frac{8}{27}.
\]

\( A \) is less likely given \( B \): If second bin is empty the first is more likely to have balls in it.
Three Card Problem

Three cards: Red/Red, Red/Black, Black/Black.
Pick one at random and place on the table. The upturned side is a Red. What is the probability that the other side is Black?
Can’t be the BB card, so... prob should be 0.5, right?

$R$: upturned card is Red; $RB$: the Red/Black card was selected.
Want $P(RB|R)$.

What’s wrong with the reasoning that leads to $\frac{1}{2}$?

$$
P(RB|R) = \frac{P(RB \cap R)}{P(R)}
$$

$$
= \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3}(1) + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3}(0)}
$$

$$
= \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}
$$

Once you are given $R$: it is twice as likely that the RR card was picked.
Gambler’s fallacy.

Flip a fair coin 51 times.
Gambler’s fallacy.

Flip a fair coin 51 times.
$A =$ “first 50 flips are heads”
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Flip a fair coin 51 times.
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Flip a fair coin 51 times.

\( A = \) “first 50 flips are heads”

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\( Pr[B|A] \) ?
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Flip a fair coin 51 times.
\( A = \) “first 50 flips are heads”
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\( A = \{HH\cdots HT, HH\cdots HH\} \)
Gambler’s fallacy.

Flip a fair coin 51 times.
A = “first 50 flips are heads”
B = “the 51st is heads”
Pr[B|A] ?

A = \{HH \cdots HT, HH \cdots HH\}
B \cap A = \{HH \cdots HH\}
Gambler’s fallacy.

Flip a fair coin 51 times.  
$A =$ “first 50 flips are heads”  
$B =$ “the 51st is heads”  
$Pr[B|A]$ ?  

$A = \{HH \cdots HT, HH \cdots HH\}$  
$B \cap A = \{HH \cdots HH\}$  

Uniform probability space.
Gambler’s fallacy.

Flip a fair coin 51 times.
$A =$ “first 50 flips are heads”
$B =$ “the 51st is heads”
$Pr[B|A]$ ?

$A = \{HH\cdots HT, HH\cdots HH\}$

$B \cap A = \{HH\cdots HH\}$

Uniform probability space.

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{2}.$
Gambler’s fallacy.

Flip a fair coin 51 times.
$A = “$first 50 flips are heads”$
$B = “the 51st is heads”$
$Pr[B|A] ?$

$A = \{HH \cdots HT, HH \cdots HH\}$
$B \cap A = \{HH \cdots HH\}$

Uniform probability space.
$\Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{2}$.

Same as $\Pr[B]$. 
Gambler’s fallacy.

Flip a fair coin 51 times.
$A =$ “first 50 flips are heads”
$B =$ “the 51st is heads”
$Pr[B|A]$ ?

$A = \{HH\cdots HT, HH\cdots HH\}$
$B \cap A = \{HH\cdots HH\}$

Uniform probability space.

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{2}.$

Same as $Pr[B]$.

The likelihood of 51st heads does not depend on the previous flips.
Product Rule

Recall the definition:
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$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$
Recall the definition:

\[ Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} \]

Hence,

\[ Pr[A \cap B] = Pr[A]Pr[B|A]. \]
Product Rule

Recall the definition:

\[
Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}. 
\]

Hence,

\[
Pr[A \cap B] = Pr[A] \cdot Pr[B|A].
\]

Consequently,

\[
Pr[A \cap B \cap C] = Pr[(A \cap B) \cap C].
\]
Product Rule

Recall the definition:

\[ Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} . \]

Hence,

\[ Pr[A \cap B] = Pr[A] Pr[B|A] . \]

Consequently,

\[ Pr[A \cap B \cap C] = Pr[(A \cap B) \cap C] = Pr[A \cap B] Pr[C|A \cap B] . \]
Product Rule

Recall the definition:

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$  

Hence,

$$Pr[A \cap B] = Pr[A]Pr[B|A].$$  

Consequently,

$$Pr[A \cap B \cap C] = Pr[(A \cap B) \cap C]$$  

$$= Pr[A \cap B]Pr[C|A \cap B]$$  

$$= Pr[A]Pr[B|A]Pr[C|A \cap B].$$
Theorem Product Rule
Let $A_1, A_2, \ldots, A_n$ be events. Then
**Theorem** Product Rule

Let $A_1, A_2, \ldots, A_n$ be events. Then

$$Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1] Pr[A_2|A_1] \cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}] .$$
Theorem Product Rule
Let \( A_1, A_2, \ldots, A_n \) be events. Then

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Proof:
Theorem Product Rule
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Proof: By induction.
Theorem Product Rule
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Proof: By induction.
Assume the result is true for $n$. 
**Theorem** Product Rule

Let $A_1, A_2, \ldots, A_n$ be events. Then

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**Proof:** By induction.

Assume the result is true for $n$. (It holds for $n = 2$.)
**Theorem** Product Rule
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**Proof:** By induction.
Assume the result is true for $n$. (It holds for $n = 2$.) Then,

$$Pr[A_1 \cap \cdots \cap A_n \cap A_{n+1}]$$
$$= Pr[A_1 \cap \cdots \cap A_n] Pr[A_{n+1} | A_1 \cap \cdots \cap A_n]$$
**Theorem** Product Rule  
Let $A_1, A_2, \ldots, A_n$ be events. Then

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\[
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= Pr[A_1] Pr[A_2 | A_1] \cdots Pr[A_n | A_1 \cap \cdots \cap A_{n-1}] Pr[A_{n+1} | A_1 \cap \cdots \cap A_n],
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**Theorem** Product Rule

Let $A_1, A_2, \ldots, A_n$ be events. Then

$$\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \Pr[A_2|A_1] \cdots \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

**Proof:** By induction. Assume the result is true for $n$. (It holds for $n = 2$.) Then,

$$\Pr[A_1 \cap \cdots \cap A_n \cap A_{n+1}]$$

$$= \Pr[A_1 \cap \cdots \cap A_n] \Pr[A_{n+1}|A_1 \cap \cdots \cap A_n]$$

$$= \Pr[A_1] \Pr[A_2|A_1] \cdots \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}] \Pr[A_{n+1}|A_1 \cap \cdots \cap A_n],$$

so that the result holds for $n+1$. \qed
Correlation

An example.

Random experiment: Pick a person at random.

Event $A$: the person has lung cancer.

Event $B$: the person is a heavy smoker.

Fact: $\Pr[A|B] = 1.17 \times \Pr[A]$.

Conclusion:

$\Rightarrow$ Smoking increases the probability of lung cancer by 17%.

$\Rightarrow$ Smoking causes lung cancer.
Correlation

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Conclusion: 
▶ Lung cancer increases the probability of smoking by 17%.
▶ Lung cancer causes smoking. Really?
Correlation


A second look.

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Note that

$$Pr[A|B] = 1.17 \times Pr[A] \iff \frac{Pr[A \cap B]}{Pr[B]} = 1.17 \times Pr[A]$$

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$$\iff Pr[A \cap B] = 1.17 \times Pr[A]Pr[B]$$

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$$Pr[A|B] = 1.17 \times Pr[A] \iff \frac{Pr[A \cap B]}{Pr[B]} = 1.17 \times Pr[A]$$
$$\iff Pr[A \cap B] = 1.17 \times Pr[A]Pr[B]$$
$$\iff Pr[B|A] = 1.17 \times Pr[B].$$

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Conclusion:

- Lung cancer increases the probability of smoking by 17%.

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Note that

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Note that

$$Pr[A|B] = 1.17 \times Pr[A] \iff \frac{Pr[A \cap B]}{Pr[B]} = 1.17 \times Pr[A]$$
$$\iff Pr[A \cap B] = 1.17 \times Pr[A]Pr[B]$$
$$\iff Pr[B|A] = 1.17 \times Pr[B].$$

Conclusion:

- Lung cancer increases the probability of smoking by 17%.
- Lung cancer causes smoking. Really?
Causality vs. Correlation

Events $A$ and $B$ are **positively correlated** if

$$Pr[A \cap B] > Pr[A]Pr[B].$$

(E.g., smoking and lung cancer.)

A and B being positively correlated does not mean that A causes B or that B causes A.

Other examples:

▶ Tesla owners are more likely to be rich. That does not mean that poor people should buy a Tesla to get rich.

▶ People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.

▶ Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?
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Total probability

Assume that $\Omega$ is the union of the disjoint sets $A_1, \ldots, A_N$. 

Indeed, $B$ is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$. 

Thus, $\Pr[B] = \Pr[A_1] \Pr[B | A_1] + \cdots + \Pr[A_N] \Pr[B | A_N]$. 
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Is your coin loaded?

Your coin is fair w.p. 1/2 or such that \( \Pr[H] = 0.6 \), otherwise.
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$Pr[B] = Pr[A \cap B] + Pr[\bar{A} \cap B] = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$

$= (1/2)(1/2) + (1/2)0.6 = 0.55.$
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$$= (1/2)(1/2) + (1/2)0.6 = 0.55.$$

Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$
Is your coin loaded?

A picture:
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Imagine 100 situations, among which

\[ m := 100 \cdot \frac{1}{2} \cdot \frac{1}{2} \] are such that \( A \) and \( B \) occur and

\[ n := 100 \cdot \frac{1}{2} \cdot 0.6 \] are such that \( \bar{A} \) and \( B \) occur.

Thus, among the \( m + n \) situations where \( B \) occurred, there are \( m \) where \( A \) occurred.

Hence,

\[
\Pr[A|B] = \frac{m}{m + n} = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) + (0.6).
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Imagine 100 situations, among which $m := 100(1/2)(1/2)$ are such that $A$ and $B$ occur and $n := 100(1/2)(0.6)$ are such that $\bar{A}$ and $B$ occur.

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$$Pr[A|B] = \frac{m}{m+n} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6}.$$
**Independence**

**Definition:** Two events $A$ and $B$ are *independent* if

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B].$$

Examples:

- When rolling two dice, $A = \text{sum is 7}$ and $B = \text{red die is 1}$ are independent;
- When rolling two dice, $A = \text{sum is 3}$ and $B = \text{red die is 1}$ are not independent;
- When flipping coins, $A = \text{coin 1 yields heads}$ and $B = \text{coin 2 yields tails}$ are independent;
- When throwing 3 balls into 3 bins, $A = \text{bin 1 is empty}$ and $B = \text{bin 2 is empty}$ are not independent;
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Independence and conditional probability

**Fact:** Two events $A$ and $B$ are **independent** if and only if
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Bayes Rule

Another picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$. 

Thus, among the $\sum m p_m q_m$ situations where $B$ occurred, there are $100 p_n q_n$ where $A_n$ occurred. Hence, $\Pr[A_n | B] = \frac{p_n q_n}{\sum m p_m q_m}$.
Bayes Rule

Another picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$. 

$$
\begin{align*}
A_1 & \\
p_1 & \rightarrow \ A_n & p_n = Pr[A_n] \\
q_1 & \rightarrow B & q_n = Pr[B | A_n] \\
p_n & \rightarrow A_n & A_1, \ldots, A_N \text{ disjoint} \\
q_n & \rightarrow B & A_1 \cup \cdots \cup A_N = \Omega \\
p_N & \rightarrow A_N \\
q_N & \rightarrow B
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Another picture: We imagine that there are $N$ possible causes $A_1, \ldots, A_N$.

Imagine 100 situations, among which $100p_nq_n$ are such that $A_n$ and $B$ occur, for $n = 1, \ldots, N$.

Thus, among the $100 \sum_m p_m q_m$ situations where $B$ occurred, there are $100p_nq_n$ where $A_n$ occurred.
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Thus, among the \( 100 \sum_m p_mq_m \) situations where \( B \) occurred, there are \( 100p_nq_n \) where \( A_n \) occurred.

Hence,

\[
Pr[A_n|B] = \frac{p_nq_n}{\sum_m p_mq_m}.
\]
Why do you have a fever?

Using Bayes' rule, we find

$$\Pr[\text{Flu} \mid \text{High Fever}] = 0.15 \times 0.80 = 0.12 \approx 0.58$$

$$\Pr[\text{Ebola} \mid \text{High Fever}] = 10^{-8} \times 0.80 = 10^{-8} \approx 5 \times 10^{-8}$$

$$\Pr[\text{Other} \mid \text{High Fever}] = 0.85 \times 0.15 \times 0.80 + 10^{-8} \times 10^{-8} \approx 0.42$$

These are the posterior probabilities. One says that 'Flu' is the Most Likely a Posteriori (MAP) cause of the high fever.
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Bayes’ Rule Operations

Bayes’ Rule is the canonical example of how information changes our opinions.
Bayes’ Rule Operations

[Environment]

Priors: $Pr[A_n]$

Observe $B$

Bayes’ Rule

Posterioris: $Pr[A_n|B]$

Conditional: $Pr[B|A_n]$

[Model of System]
Bayes’ Rule is the canonical example of how information changes our opinions.
Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born c. 1701
London, England

Died 7 April 1761 (aged 59)
Tunbridge Wells, Kent, England

Residence Tunbridge Wells, Kent, England

Nationality English

Known for Bayes' theorem

A Bayesian picture of Thomas Bayes.
Testing for disease.

Let's watch TV!!

Random Experiment: Pick a random male.

Outcomes:

A - prostate cancer.
B - positive PSA test.

$\Pr[A] = 0.0016$, (0.16% of the male population is affected.)

$\Pr[B|A] = 0.80$ (80% chance of positive test with disease.)

$\Pr[B|A] = 0.10$ (10% chance of positive test without disease.)


Positive PSA test (B). Do I have disease? $\Pr[A|B]$??
Testing for disease.

Let’s watch TV!!
Random Experiment: Pick a random male.
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Outcomes: *(test, disease)*
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\(Pr[A|B]??)\)
Using Bayes' rule, we find
\[ P[A|B] = 0.0016 \times 0.80 + 0.9984 \times 0.10 = 0.013. \]

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence...

Death.
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Summary

Events, Conditional Probability, Independence, Bayes’ Rule
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Key Ideas:

- **Conditional Probability:**
  \[ Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} \]
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- Conditional Probability:
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- Independence: \( Pr[A \cap B] = Pr[A]Pr[B] \).
Events, Conditional Probability, Independence, Bayes’ Rule

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- **Bayes’ Rule:**
  \[ Pr[A_n|B] = \frac{Pr[A_n]Pr[B|A_n]}{\sum_m Pr[A_m]Pr[B|A_m]}. \]
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Key Ideas:

- Conditional Probability:
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\[ Pr[A_n|B] = \text{posterior probability}; \ Pr[A_n] = \text{prior probability} . \]
Summary

Events, Conditional Probability, Independence, Bayes’ Rule

Key Ideas:

- **Conditional Probability:**
  \[ Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} \]

- **Independence:**
  \[ Pr[A \cap B] = Pr[A]Pr[B] \]

- **Bayes’ Rule:**
  \[
  Pr[A_n|B] = \frac{Pr[A_n]Pr[B|A_n]}{\sum_m Pr[A_m]Pr[B|A_m]}. 
  \]
  
  \( Pr[A_n|B] = \) posterior probability; \( \Pr[A_n] = \) prior probability

- All these are possible:
  \[ Pr[A|B] < \Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = \Pr[A] \]