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A bit more review of discrete math.

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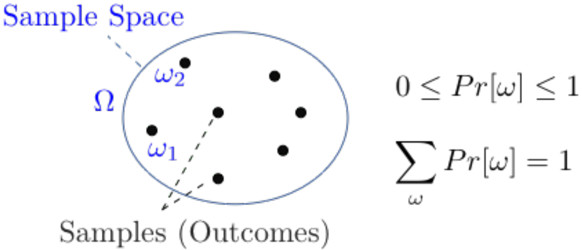

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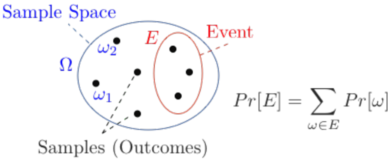

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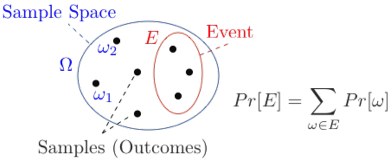

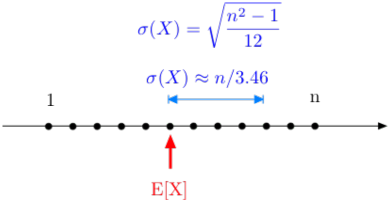

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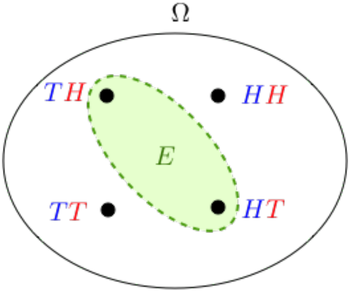

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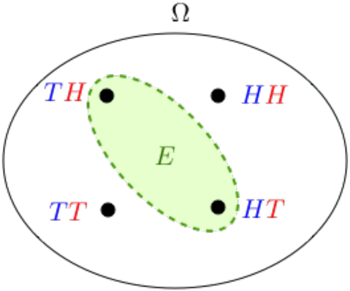

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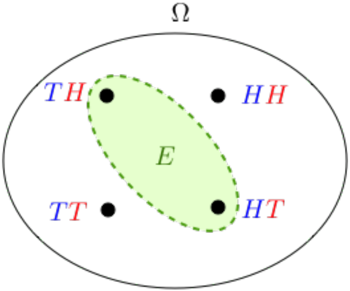

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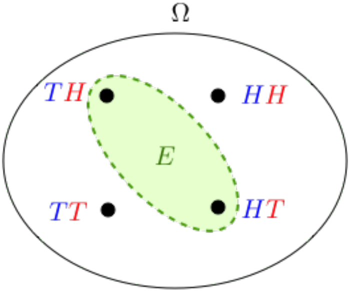

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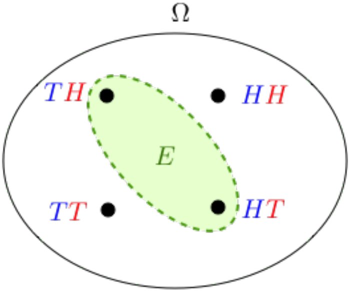

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(a) $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$;

(inclusion-exclusion property)

(b) $Pr[A_1 \cup \cdots \cup A_n] \leq Pr[A_1] + \cdots + Pr[A_n]$;

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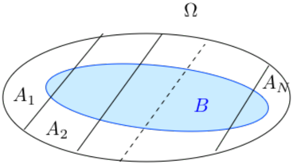

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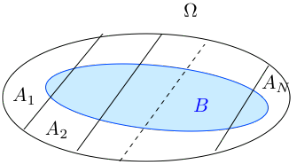

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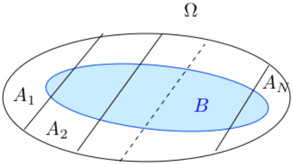

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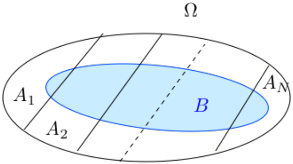

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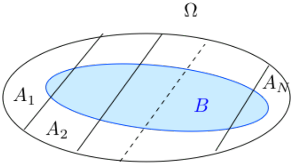

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Adding up probability of them, get $Pr[\omega]$ in sum.

Conditional Probability.

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$

Yet more fun with conditional probability.

Toss a red and a blue die, sum is 7,
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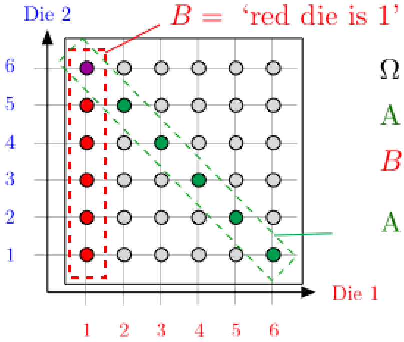
Ω : Uniform



$\Omega = \{1, \ldots, 6\}^2$

A $= \{(1, 6), \ldots, (6,1)\}$

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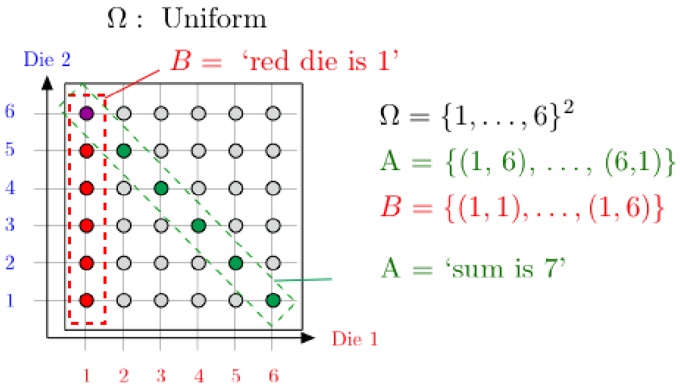

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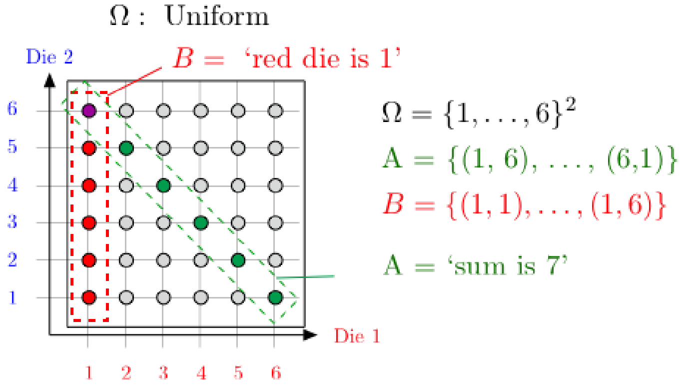

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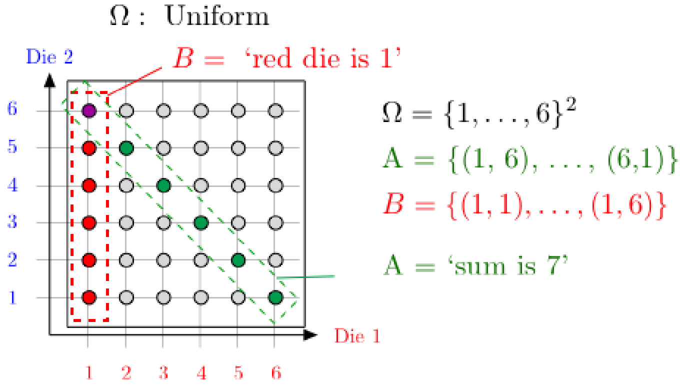

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Observing A does not change your mind about the likelihood of B.

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so that the result holds for $n+1$. $\square$

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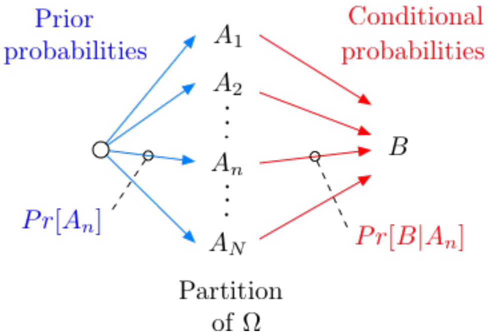

$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_N]Pr[B|A_N].$$

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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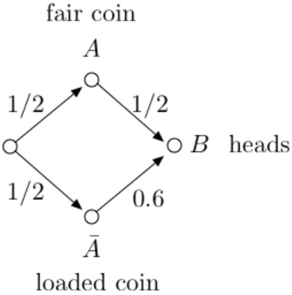

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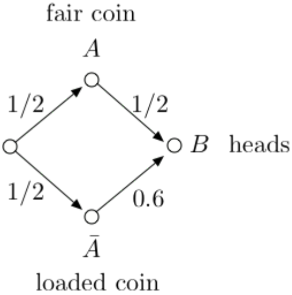

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Independence and conditional probability

Fact: Two events A and B are **independent** if and only if

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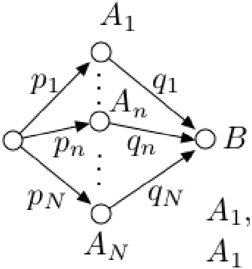

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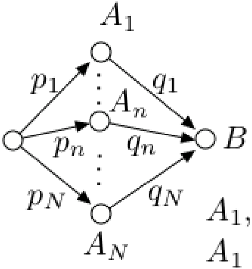

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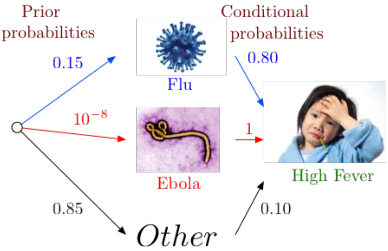

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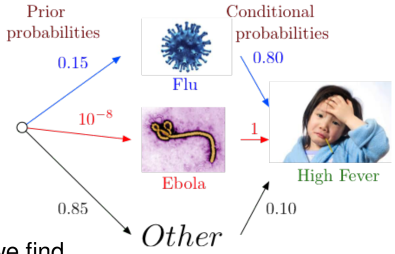

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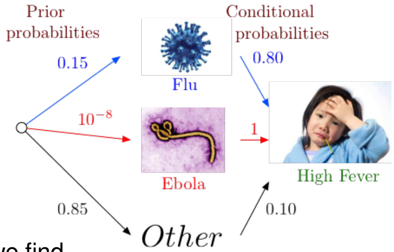

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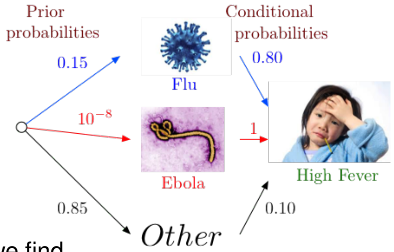

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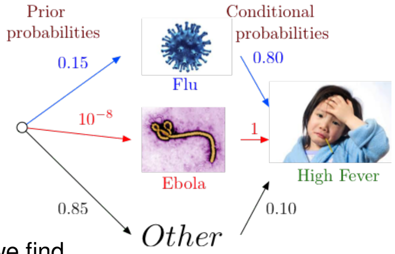

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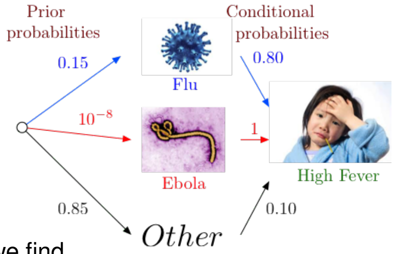

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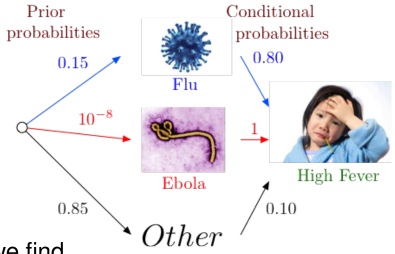

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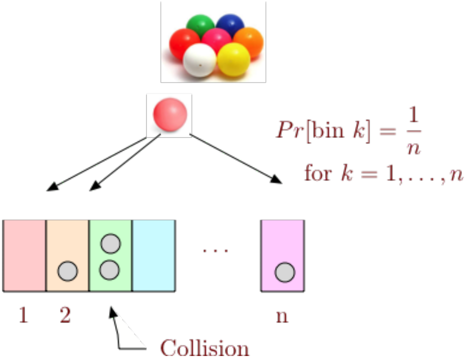

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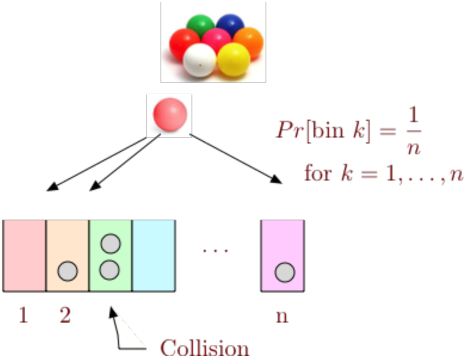

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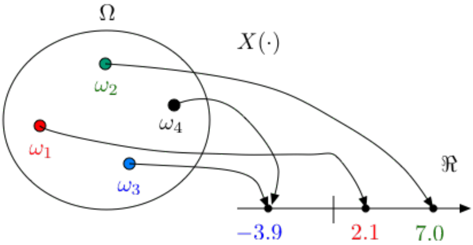

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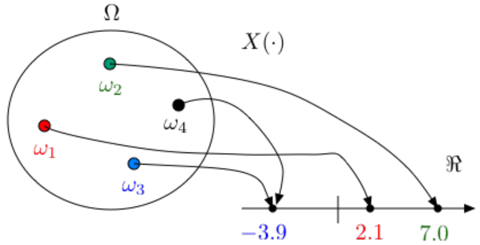

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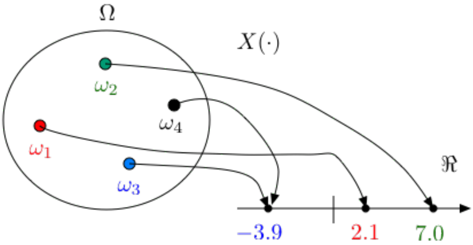

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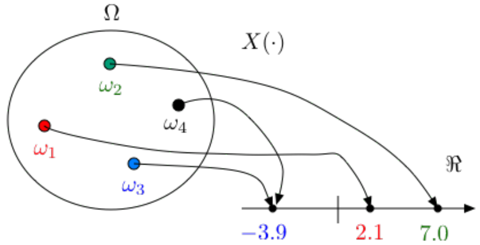

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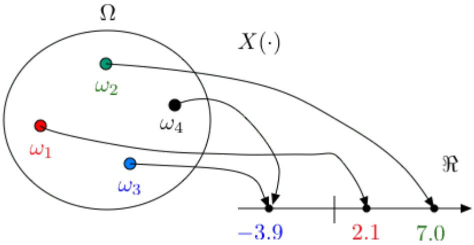

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What varies at random (from experiment to experiment)?

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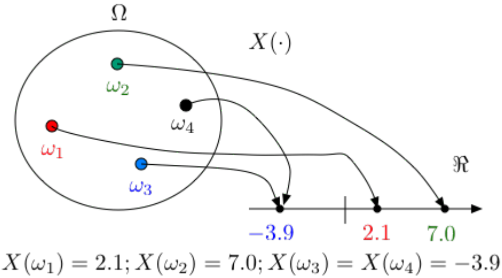

The function $X(\cdot)$ is defined on the outcomes Ω.

The function $X(\cdot)$ is not random, not a variable!

What varies at random (from experiment to experiment)? The outcome!

Number of pips in two dice.

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“What is the likelihood of getting n pips?”

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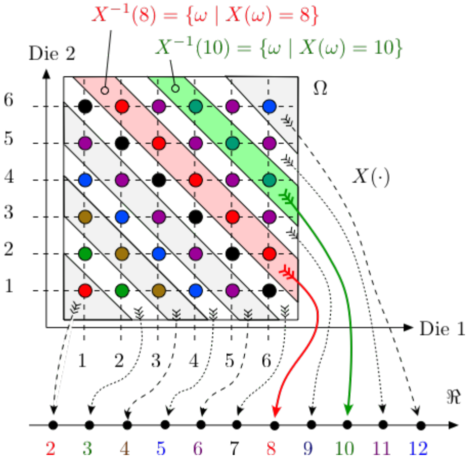
"What is the likelihood of getting *n* pips?"



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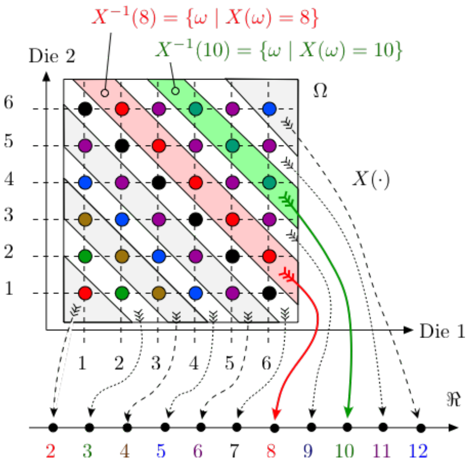

$Pr[X = 10] =$

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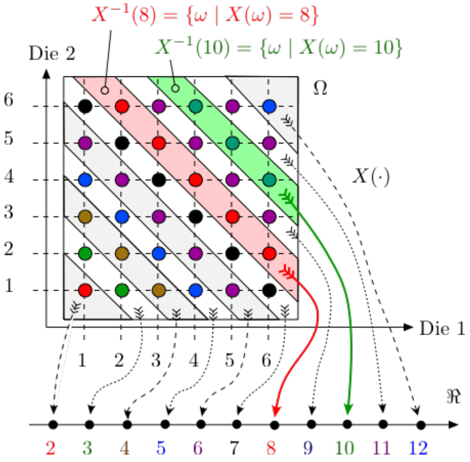

$Pr[X = 10] = 3/36 =$

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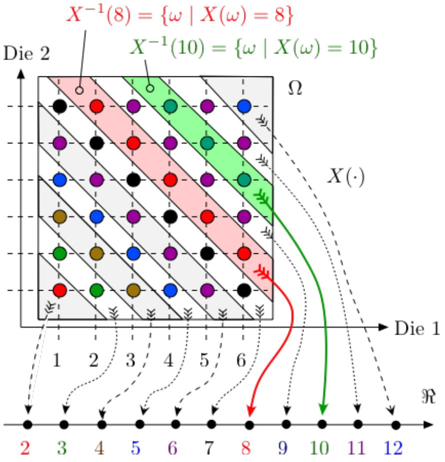

$$Pr[X = 10] = 3/36 = Pr[X^{-1}(10)];$$

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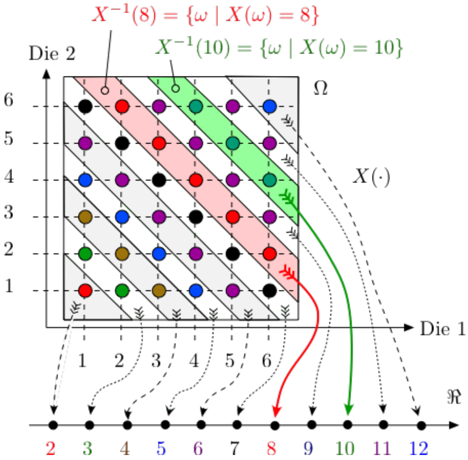

$Pr[X = 10] = 3/36 = Pr[X^{-1}(10)]; Pr[X = 8] =$

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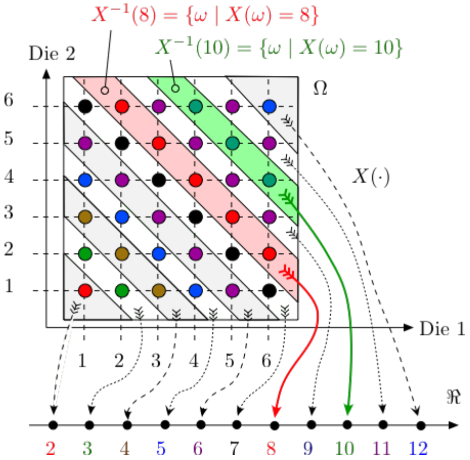

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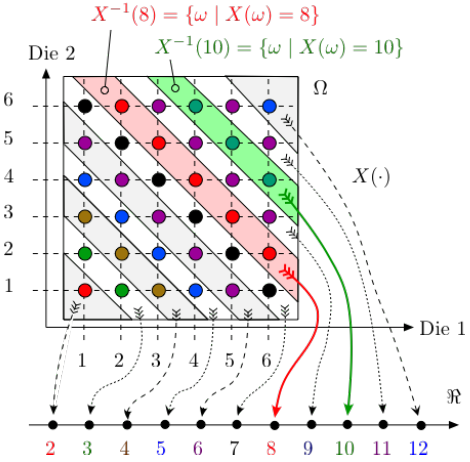

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The probability of X taking on a value a.

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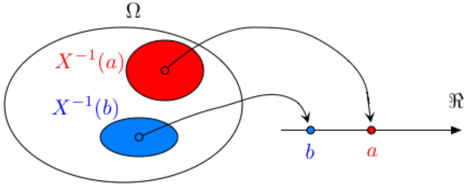

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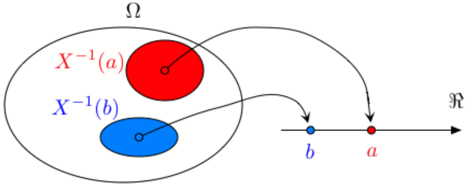

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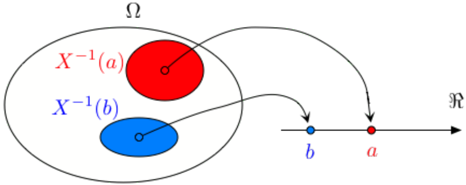

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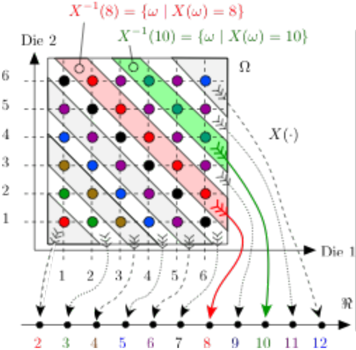

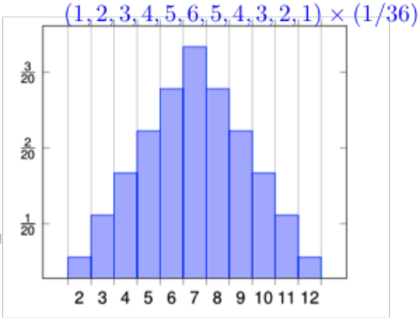

Named Distributions.

Some distributions come up over and over again.

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Let’s cover one for this review.

The binomial distribution.

Flip n coins with heads probability p.

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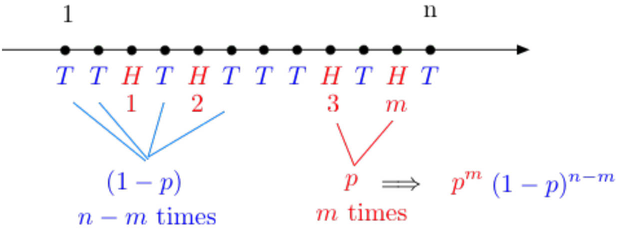
Probability of "$X = i$" is sum of $Pr[\omega]$, $\omega \in$ "$X = i$".

$$Pr[X = i] = \binom{n}{i} p^i(1-p)^{n-i}, i = 0, 1, \ldots, n : B(n,p) \text{distribution}$$

The binomial distribution.



$\binom{n}{m}$ outcomes with m Hs and $n-m$ Ts

$$\Longrightarrow \boxed{Pr[X=m] = \binom{n}{m} p^m (1-p)^{n-m}}$$

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- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.

Discrete Math:Review

Modular Arithmetic Inverses and GCD

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by adding and subtracting multiples of x and y

Non-recursive extended gcd.

Example: $p = 7$, $q = 11$.

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$d = e^{-1} = -17 = 43 = \pmod{60}$

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$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \mod p,$$

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Proof: Consider $T = \{a \cdot 1 \pmod{p}, \ldots, a \cdot (p-1) \pmod{p}\}$.

T is range of function $f(x) = ax \mod (p)$ for set $S = \{1, \ldots, p-1\}$.
Invertible function: one-to-one.
$T \subseteq S$ since $0 \notin T$.
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Product of elts of T = Product of elts of S.

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RSA, Public Key, and Signatures.

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Verify: Check $C = E(C)$.

$E(D(C,k),K) = (C^d)^e = C \pmod N$

Simple Chinese Remainder Theorem.

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My love is won.

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Only solution?

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$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.

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Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod{m}$ and $(x-y) \equiv 0 \pmod{n}$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn$

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

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CRT Thm: Unique solution $\pmod{mn}$.
Proof:
Consider $u = n(n^{-1} \pmod{m})$.
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For each modulus n_i,
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Polynomials

Property 1: Any degree d polynomial over a field has at most d roots.

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Property 2 and pigeonhole principle.

Welsh-Berlekamp

Idea: Error locator polynomial of degree k with zeros at errors.

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Counting

First Rule

Counting

First Rule
Second Rule

Counting

First Rule
Second Rule
Stars/Bars

Counting

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Stars/Bars
Common Scenarios: Sampling, Balls in Bins.

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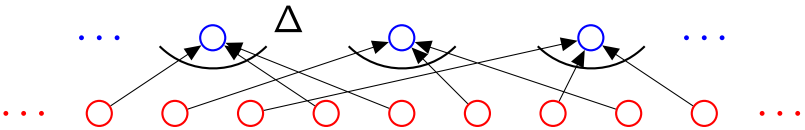
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$**. Product Rule.**
Second rule: when order doesn't matter divide..when possible.



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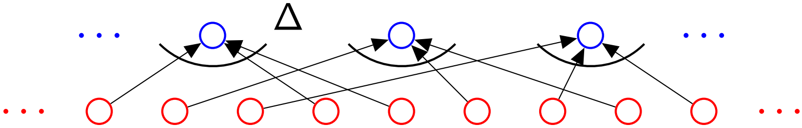

3 card Poker deals: 52

Example: visualize.

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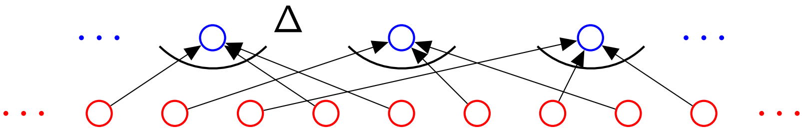

3 card Poker deals: 52×51

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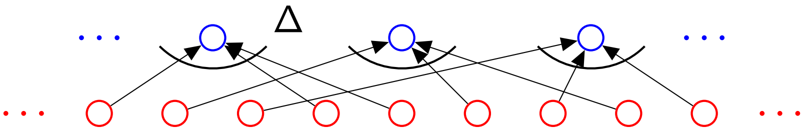

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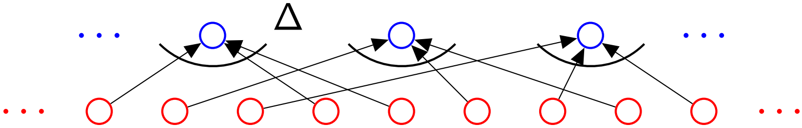

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$.

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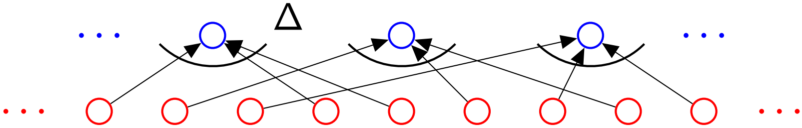

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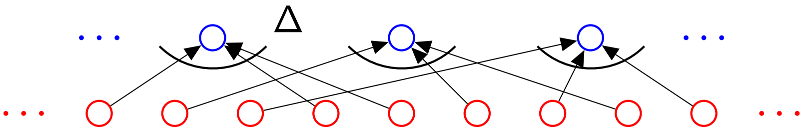

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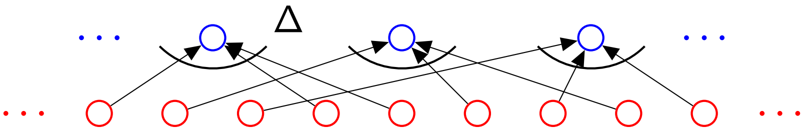

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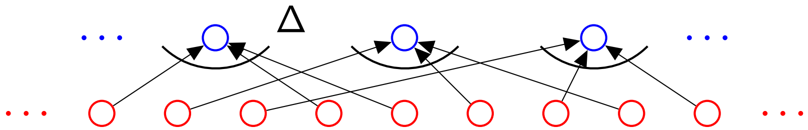

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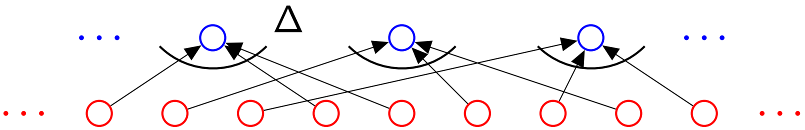

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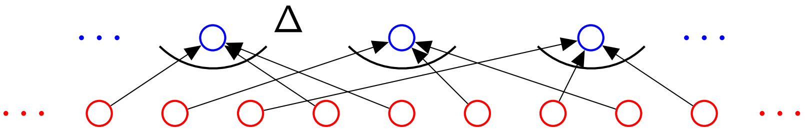

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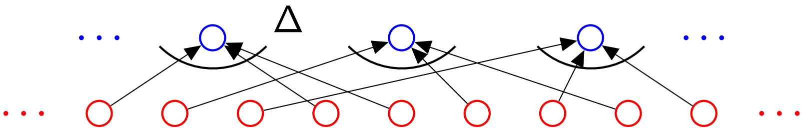

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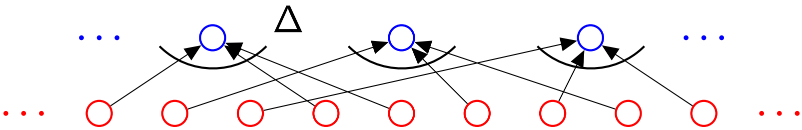

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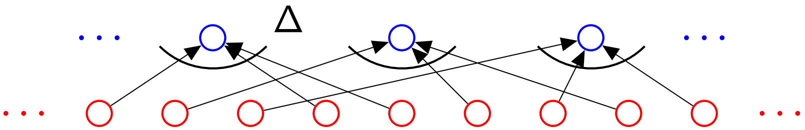

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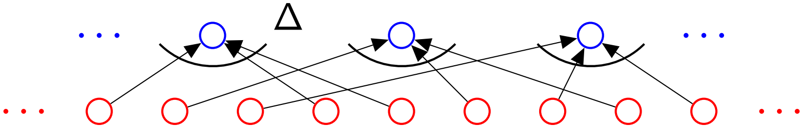

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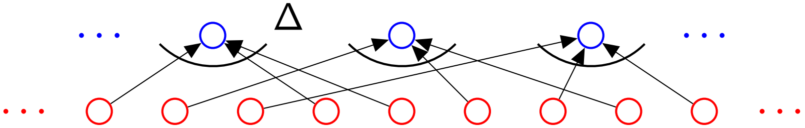

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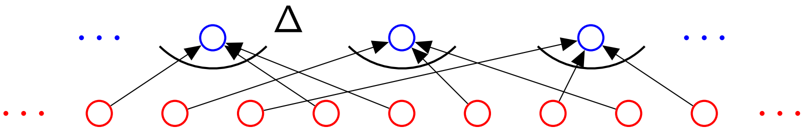

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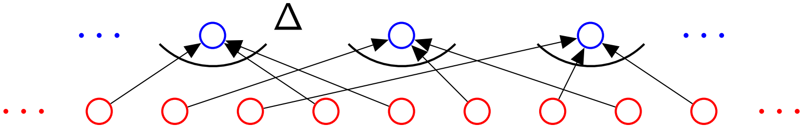

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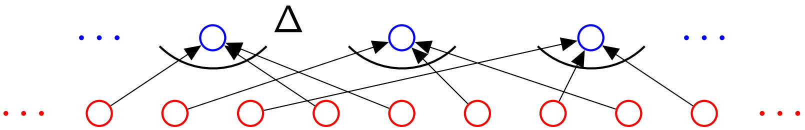

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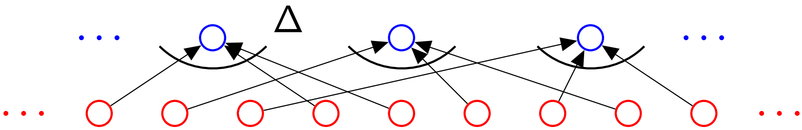

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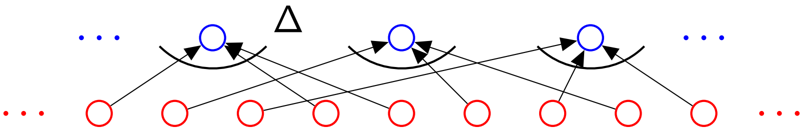

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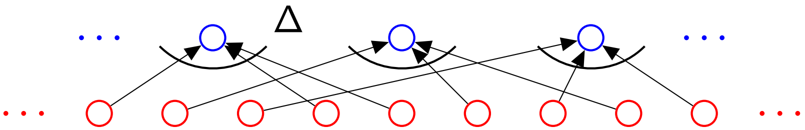

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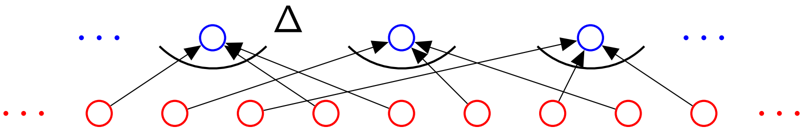

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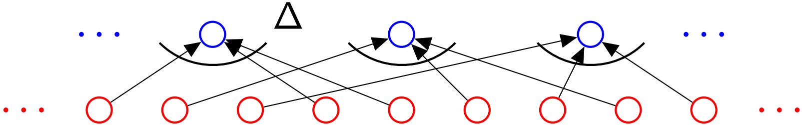

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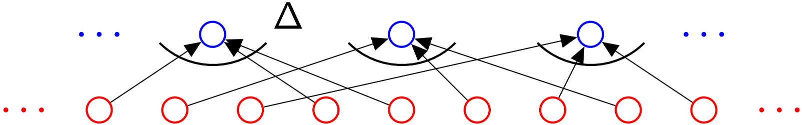

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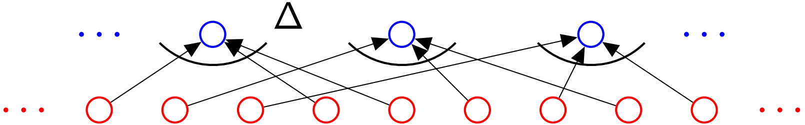

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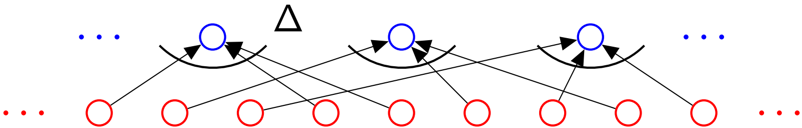

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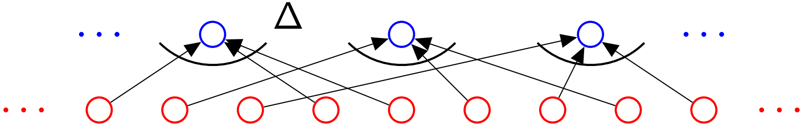

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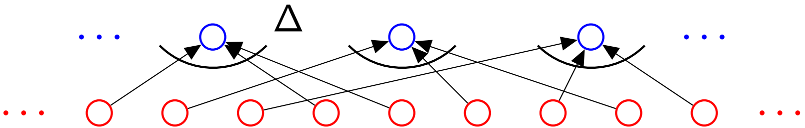

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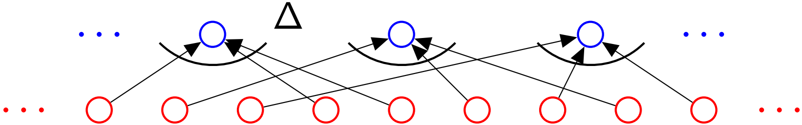

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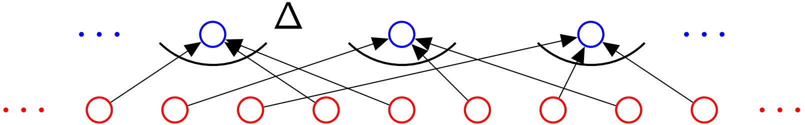

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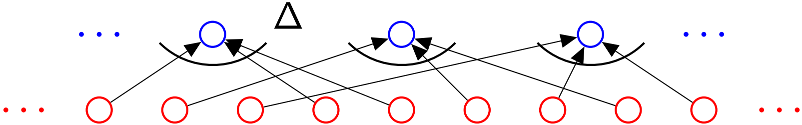

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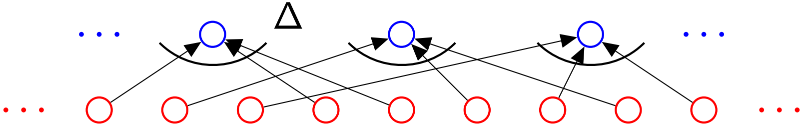

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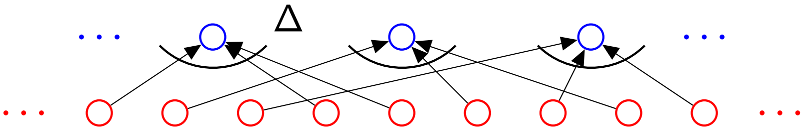

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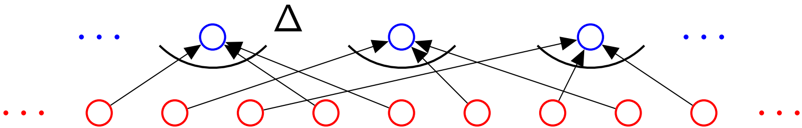

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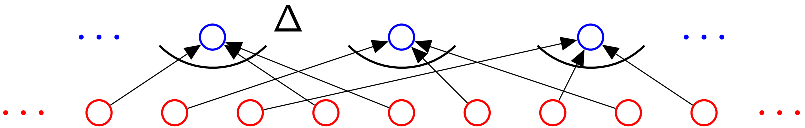

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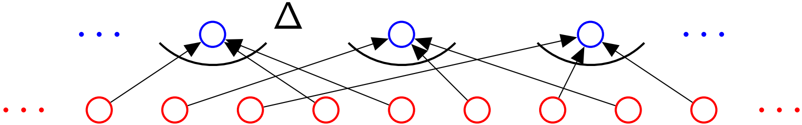

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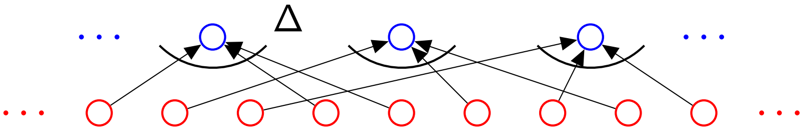

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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

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Isomporphism principle.

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Example.

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Given a function, $f : D \rightarrow R$.

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$[0,1]$ is same cardinality as nonnegative reals!

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All countably infinite sets are the same cardinality as each other.

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Diagonalization: power set of Integers.

The set of all subsets of N.

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Theorem: The set of all subsets of N is not countable.

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If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

D is a subset of N.

L does not contain all subsets of N.

Contradiction.

Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

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Halting problem is undecibable.

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Time: approximately 120 minutes.

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Some longer questions.

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Wrapup.

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Watch Piazza for Logistics!

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Other issues....

Wrapup.

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