CS70: Random Variables (contd.)

Random Variables: Expectation

1. Random Variables: Brief Review
2. Expectation and properties
3. Important Distributions

Random Variables: Definitions

Definition

A random variable, X, for a random experiment with sample space Ω is a function $X : \Omega \to \Re$.

Thus, $X(\cdot)$ assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

Definitions

(a) For $a \in \Re$, one defines

$$X^{-1}(a) := \{\omega \in \Omega \mid X(\omega) = a\}.$$

(b) For $A \subset \Re$, one defines

$$X^{-1}(A) := \{\omega \in \Omega \mid X(\omega) \in A\}.$$

(c) The probability that $X = a$ is defined as

$$Pr[X = a] = Pr[X^{-1}(a)].$$

(d) The probability that $X \in A$ is defined as

$$Pr[X \in A] = Pr[X^{-1}(A)].$$

(e) The distribution of a random variable X, is

$$\{(a, Pr[X = a]) : a \in \mathscr{A}\},$$

where $\mathscr{A}$ is the *range* of X. That is, $\mathscr{A} = \{X(\omega), \omega \in \Omega\}$.

Expectation - Definition

Definition: The **expected value** (or mean, or expectation) of a random variable X is

$$E[X] = \sum_a a \times Pr[X = a].$$

Theorem:

$$E[X] = \sum_\omega X(\omega) \times Pr[\omega].$$

An Example

Flip a fair coin three times.

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$X =$ number of H's: $\{3,2,2,2,1,1,1,0\}$.

Thus,

$$\sum_\omega X(\omega) Pr[\omega] = \{3+2+2+2+1+1+1+0\} \times \frac{1}{8}.$$

Also,

$$\sum_a a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

Win or Lose.

Expected winnings for heads/tails games, with 3 flips?

Recall the definition of the random variable X:

$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \to \{3, 1, 1, -1, 1, -1, -1, -3\}$.

$$E[X] = 3 \times \frac{1}{8} + 1 \times \frac{3}{8} - 1 \times \frac{3}{8} - 3 \times \frac{1}{8} = 0.$$

Can you ever win 0?

Apparently: expected value is not a common value, by any means.

The expected value of X is not the value that you expect!

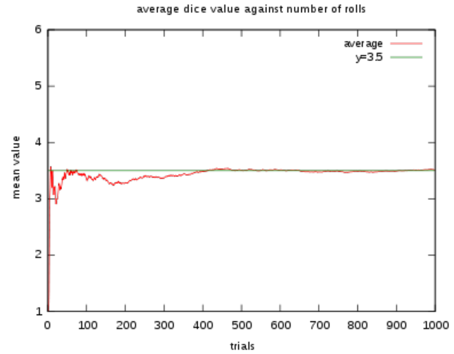
It is the average value per experiment, if you perform the experiment many times:

$$\frac{X_1 + \cdots + X_n}{n}, \text{ when } n \gg 1.$$

The fact that this average converges to $E[X]$ is a theorem: the Law of Large Numbers. (See later.)

Law of Large Numbers

An Illustration: Rolling Dice



Indicators

Definition
Let A be an event. The random variable X defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event A.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$.

Linearity of Expectation

Theorem: Expectation is linear

$$E[a_1X_1 + \cdots + a_nX_n] = a_1E[X_1] + \cdots + a_nE[X_n].$$

Proof:

$$\begin{aligned} &E[a_1X_1 + \cdots + a_nX_n] \\ &= \sum_\omega (a_1X_1 + \cdots + a_nX_n)(\omega)Pr[\omega] \\ &= \sum_\omega (a_1X_1(\omega) + \cdots + a_nX_n(\omega))Pr[\omega] \\ &= a_1\sum_\omega X_1(\omega)Pr[\omega] + \cdots + a_n\sum_\omega X_n(\omega)Pr[\omega] \\ &= a_1E[X_1] + \cdots + a_nE[X_n]. \end{aligned}$$

$\square$

Note: If we had defined $Y = a_1X_1 + \cdots + a_nX_n$ has had tried to compute $E[Y] = \sum_y yPr[Y = y]$, we would have been in trouble!

Using Linearity - 1: Pips (dots) on dice

Roll a die n times.

X_m = number of pips on roll m.

$X = X_1 + \cdots + X_n$ = total number of pips in n rolls.

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_n] \\ &= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\ &= nE[X_1], \text{ because the } X_m \text{ have the same distribution} \end{aligned}$$

Now,

$$E[X_1] = 1 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{6 \times 7}{2} \times \frac{1}{6} = \frac{7}{2}.$$

Hence,

$$E[X] = \frac{7n}{2}.$$

Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

Using Linearity - 2: Random assignments Example

Hand out assignments at random to n students.

X = number of students that get their own assignment back.

$X = X_1 + \cdots + X_n$ where
$X_m = 1\{\text{student } m \text{ gets his/her own assignment back}\}$.

One has

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_n] \\ &= E[X_1] + \cdots + E[X_n], \text{ by linearity} \\ &= nE[X_1], \text{ because all the } X_m \text{ have the same distribution} \\ &= nPr[X_1 = 1], \text{ because } X_1 \text{ is an indicator} \\ &= n(1/n), \text{ because student 1 is equally likely} \\ &\qquad \text{to get any one of the } n \text{ assignments} \\ &= 1. \end{aligned}$$

Note that linearity holds even though the X_m are not independent.

Note: What is $Pr[X = m]$? Tricky ....

Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p. X - number of heads

Binomial Distibution: $Pr[X = i]$, for each i.

$$Pr[X = i] = \binom{n}{i} p^i(1-p)^{n-i}.$$

$$E[X] = \sum_i i \times Pr[X = i] = \sum_i i \times \binom{n}{i} p^i(1-p)^{n-i}.$$

Uh oh. ... Or... a better approach: Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = 1 \times Pr[\text{"heads"}] + 0 \times Pr[\text{"tails"}] = p.$

Moreover $X = X_1 + \cdots X_n$ and

$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i] = np.$

Calculating $E[g(X)]$

Let $Y = g(X)$. Assume that we know the distribution of X.

We want to calculate $E[Y]$.

Method 1: We calculate the distribution of Y:

$$Pr[Y = y] = Pr[X \in g^{-1}(y)] \text{ where } g^{-1}(x) = \{x \in \Re : g(x) = y\}.$$

This is typically rather tedious!

Method 2: We use the following result.

Theorem:
$$E[g(X)] = \sum_x g(x)Pr[X = x].$$

Proof:

$$\begin{aligned} E[g(X)] &= \sum_\omega g(X(\omega))Pr[\omega] = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega))Pr[\omega] \\ &= \sum_x \sum_{\omega \in X^{-1}(x)} g(x)Pr[\omega] = \sum_x g(x) \sum_{\omega \in X^{-1}(x)} Pr[\omega] \\ &= \sum_x g(x)Pr[X = x]. \end{aligned}$$

$\square$

An Example

Let X be uniform in $\{-2,-1,0,1,2,3\}$.

Let also $g(X) = X^2$. Then (method 2)

$$
\begin{aligned}
E[g(X)] &= \sum_{x=-2}^{3} x^2 \frac{1}{6} \\
&= \{4+1+0+1+4+9\}\frac{1}{6} = \frac{19}{6}.
\end{aligned}
$$

Method 1 - We find the distribution of $Y = X^2$:

$$
Y = \begin{cases} 4, & \text{w.p. } \frac{2}{6} \\ 1, & \text{w.p. } \frac{2}{6} \\ 0, & \text{w.p. } \frac{1}{6} \\ 9, & \text{w.p. } \frac{1}{6}. \end{cases}
$$

Thus,

$$
E[Y] = 4\frac{2}{6} + 1\frac{2}{6} + 0\frac{1}{6} + 9\frac{1}{6} = \frac{19}{6}.
$$

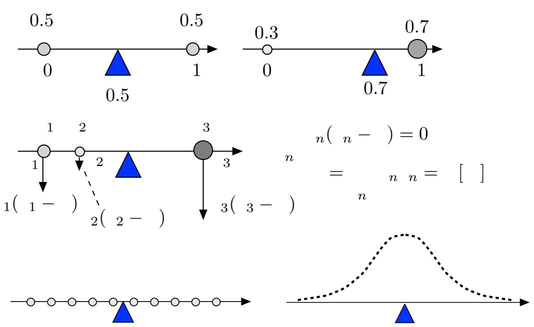
Summary

Random Variables

- A random variable X is a function $X : \Omega \to \Re$.
- $Pr[X = a] := Pr[X^{-1}(a)] = Pr[\{\omega \mid X(\omega) = a\}]$.
- $Pr[X \in A] := Pr[X^{-1}(A)]$.
- The distribution of X is the list of possible values and their probability: $\{(a, Pr[X = a]), a \in \mathscr{A}\}$.
- $E[X] := \sum_a aPr[X = a]$.
- Expectation is Linear.

Center of Mass

The expected value has a *center of mass* interpretation:



Monotonicity

Definition
Let X, Y be two random variables on Ω. We write $X \le Y$ if $X(\omega) \le Y(\omega)$ for all $\omega \in \Omega$, and similarly for $X \ge Y$ and $X \ge a$ for some constant a.

Facts
(a) If $X \ge 0$, then $E[X] \ge 0$.
(b) If $X \le Y$, then $E[X] \le E[Y]$.

Proof
(a) If $X \ge 0$, every value a of X is nonnegative. Hence,

$$
E[X] = \sum_a aPr[X = a] \ge 0.
$$

(b) $X \le Y \Rightarrow Y - X \ge 0 \Rightarrow E[Y] - E[X] = E[Y - X] \ge 0$. $\square$

Example:

$$
B = \cup_m A_m \Rightarrow 1_B(\omega) \le \sum_m 1_{A_m}(\omega) \Rightarrow Pr[\cup_m A_m] \le \sum_m Pr[A_m].
$$