CS70: Random Variables (contd.)

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Random Variables: Expectation

1. Random Variables: Brief Review
2. Expectation and properties
3. Important Distributions

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Also,

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the Law of Large Numbers. (See later.)

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An Illustration: Rolling Dice

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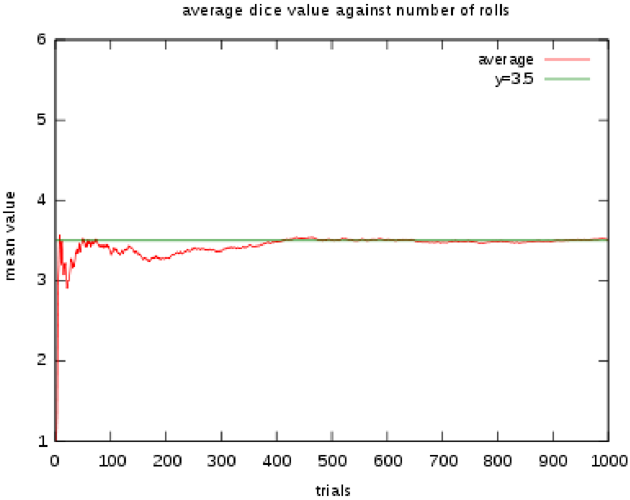

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Thus, we will write $X = 1_A$.

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$\square$

Note: If we had defined $Y=a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y]=\sum_y yPr[Y=y]$, we would have been in trouble!

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$$E[X_1] = 1 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = \frac{6 \times 7}{2} \times \frac{1}{6} = \frac{7}{2}.$$

Hence,

$$E[X] = \frac{7n}{2}.$$

Using Linearity - 1: Pips (dots) on dice

Roll a die n times.

X_m = number of pips on roll m.

$X = X_1 + \cdots + X_n$ = total number of pips in n rolls.

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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Hand out assignments at random to n students.

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Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p.

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Flip n coins with heads probability p. X - number of heads

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$\square$

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Method 1 - We find the distribution of $Y = X^2$:

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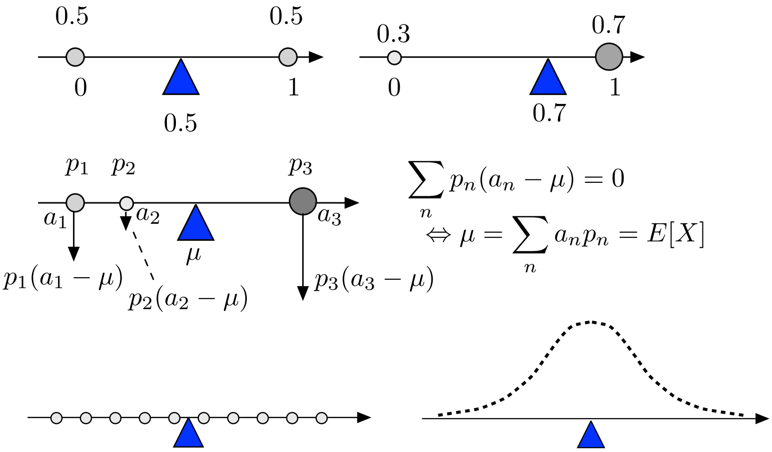

$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$

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