CS70: Random Variables (contd.)

Important Distributions

1. Expectation: Brief Review
2. Important Distributions: Binomial, Uniform, Geometric, Poisson

Review: Expectation

- $E[X] := \sum_x xPr[X = x] = \sum_\omega X(\omega)Pr[\omega]$.
- $E[g(X)] = \sum_x g(x)Pr[X = x]$
  $= \sum_\omega g(X(\omega))Pr[\omega]$
- $E[aX + bY + c] = aE[X] + bE[Y] + c$.

Linearity of Expectation

Theorem: Expectation is linear

$$E[a_1X_1+\cdots+a_nX_n] = a_1E[X_1]+\cdots+a_nE[X_n].$$

Proof:

$$\begin{aligned}
&E[a_1X_1+\cdots+a_nX_n]\\
&\quad = \sum_{\omega}(a_1X_1+\cdots+a_nX_n)(\omega)Pr[\omega]\\
&\quad = \sum_{\omega}(a_1X_1(\omega)+\cdots+a_nX_n(\omega))Pr[\omega]\\
&\quad = a_1\sum_{\omega}X_1(\omega)Pr[\omega]+\cdots+a_n\sum_{\omega}X_n(\omega)Pr[\omega]\\
&\quad = a_1E[X_1]+\cdots+a_nE[X_n].
\end{aligned}$$

$\square$

Note: If we had defined $Y = a_1X_1+\cdots+a_nX_n$ has had tried to compute $E[Y] = \sum_y yPr[Y=y]$, we would have been in trouble!

Using Linearity: Binomial Distribution.

Flip n coins with heads probability p. X - number of heads

Binomial Distibution: $Pr[X=i]$, for each i.

$$Pr[X=i] = \binom{n}{i} p^i (1-p)^{n-i}.$$

$$E[X] = \sum_i i \times Pr[X=i] = \sum_i i \times \binom{n}{i} p^i (1-p)^{n-i}.$$

Uh oh. ... Or... a better approach: Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

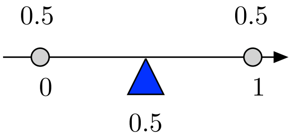
$E[X_i] = 1 \times Pr[\text{``}heads''] + 0 \times Pr[\text{``}tails''] = p.$

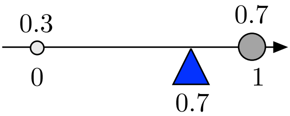
Moreover $X = X_1 + \cdots X_n$ and

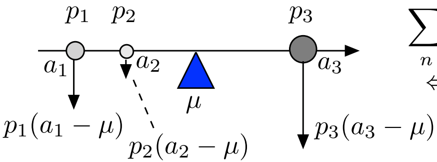
$E[X] = E[X_1] + E[X_2] + \cdots E[X_n] = n \times E[X_i] = np.$

Center of Mass

The expected value has a *center of mass* interpretation:

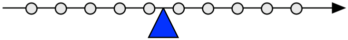


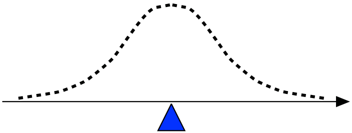




$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$





Indicator Random Variable

Definition

Let A be an event. The random variable X defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event A.

Note that $Pr[X=1] = Pr[A]$ and $Pr[X=0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X=1] + 0 \times Pr[X=0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$.

Uniform Distribution

Roll a six-sided balanced die. Let X be the number of pips (dots). Then X is equally likely to take any of the values $\{1,2,\ldots,6\}$. We say that X is *uniformly distributed* in $\{1,2,\ldots,6\}$.

More generally, we say that X is uniformly distributed in $\{1,2,\ldots,n\}$ if $Pr[X = m] = 1/n$ for $m = 1,2,\ldots,n$.
In that case,

$$E[X] = \sum_{m=1}^{n} mPr[X = m] = \sum_{m=1}^{n} m \times \frac{1}{n} = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Geometric Distribution

Let's flip a coin with $Pr[H] = p$ until we get H.

For instance:

$$\begin{aligned}\omega_1 &= H, \text{ or}\\ \omega_2 &= T\ H, \text{ or}\\ \omega_3 &= T\ T\ H, \text{ or}\\ \omega_n &= T\ T\ T\ T\ \cdots\ T\ H.\end{aligned}$$

Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

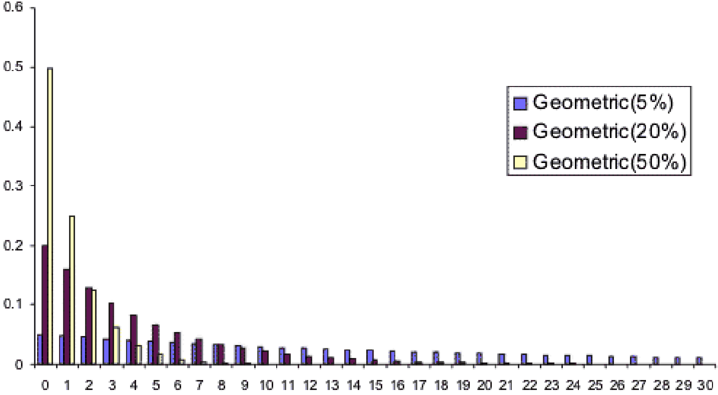
Let X be the number of flips until the first H. Then, $X(\omega_n) = n$.

Also,

$$Pr[X = n] = (1-p)^{n-1}p, \; n \geq 1.$$

Geometric Distribution

$$Pr[X = n] = (1 - p)^{n-1} p, n \geq 1.$$



Geometric Distribution

$$Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

Note that

$$\sum_{n=1}^{\infty} Pr[X_n] = \sum_{n=1}^{\infty} (1-p)^{n-1}p = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p\sum_{n=0}^{\infty}(1-p)^n.$$

Now, if $|a| < 1$, then $S := \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$. Indeed,

$$\begin{aligned} S &= 1 + a + a^2 + a^3 + \cdots \\ aS &= \quad\;\; a + a^2 + a^3 + a^4 + \cdots \\ (1-a)S &= 1 + a - a + a^2 - a^2 + \cdots = 1. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} Pr[X_n] = p\,\frac{1}{1-(1-p)} = 1.$$

Geometric Distribution: Expectation

$$X =_D G(p), \text{ i.e., } Pr[X = n] = (1-p)^{n-1}p, n \geq 1.$$

One has

$$E[X] = \sum_{n=1}^{\infty} nPr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Thus,

$$\begin{aligned}
E[X] &= p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \\
(1-p)E[X] &= (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \\
pE[X] &= p + (1-p)p + (1-p)^2p + (1-p)^3p + \cdots \\
&\quad \text{by subtracting the previous two identities} \\
&= \sum_{n=1}^{\infty} Pr[X = n] = 1.
\end{aligned}$$

Hence,

$$E[X] = \frac{1}{p}.$$

Coupon Collectors Problem.

Experiment: Get coupons at random from n until collect all n coupons.
Outcomes: $\{123145...,56765...\}$
Random Variable: X - length of outcome.

$E[X]$=?

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second coupon after getting first.

Pr[“get second coupon”|“got ~~milk~~ first coupon”] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith coupon|“got $i-1$rst coupons”] $= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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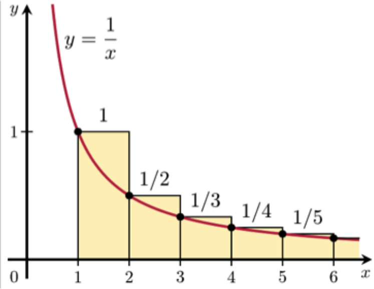


A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Geometric Distribution: Memoryless

Let X be $G(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

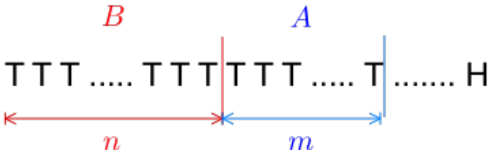
$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

Proof:

$$\begin{aligned}
Pr[X > n+m | X > n] &= \frac{Pr[X > n+m \text{ and } X > n]}{Pr[X > n]} \\
&= \frac{Pr[X > n+m]}{Pr[X > n]} \\
&= \frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m \\
&= Pr[X > m].
\end{aligned}$$

Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n + m | X > n] = Pr[X > m], m, n \geq 0.$$



$$Pr[X > n + m | X > n] = Pr[A|B] = Pr[A] = Pr[X > m].$$

The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0,1,2,\ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

[See later for a proof.]

If $X = G(p)$, then $Pr[X \geq i] = Pr[X > i-1] = (1-p)^{i-1}$.

Hence,

$$E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^{i} = \frac{1}{1-(1-p)} = \frac{1}{p}.$$

Expected Value of Integer RV

Theorem: For a r.v. X that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

Proof: One has

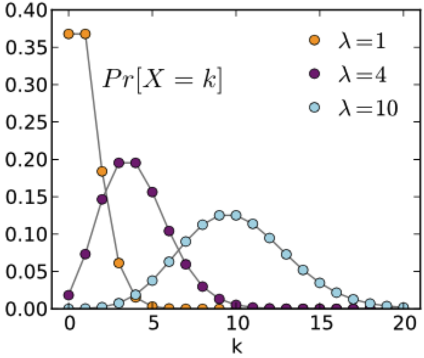
$$\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} i \times Pr[X = i] \\
&= \sum_{i=1}^{\infty} i\{Pr[X \geq i] - Pr[X \geq i+1]\} \\
&= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - i \times Pr[X \geq i+1]\} \\
&= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - (i-1) \times Pr[X \geq i]\} \\
&= \sum_{i=1}^{\infty} Pr[X \geq i].
\end{aligned}$$

□

Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.

Poisson Distribution is distribution of X "for large n."



Poisson

Experiment: flip a coin n times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: X - number of heads. Thus, $X = B(n, \lambda/n)$.
Poisson Distribution is distribution of X "for large n."
We expect $X \ll n$. For $m \ll n$ one has

$$\begin{aligned}
Pr[X = m] &= \binom{n}{m} p^m (1-p)^{n-m}, \text{ with } p = \lambda/n \\
&= \frac{n(n-1)\cdots(n-m+1)}{m!} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&= \frac{n(n-1)\cdots(n-m+1)}{n^m} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&\approx^{(1)} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \approx \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^n \approx^{(2)} \frac{\lambda^m}{m!} e^{-\lambda}.
\end{aligned}$$

For (1) we used $m \ll n$; for (2) we used $(1 - a/n) \approx e^{-a/n}$ for $a/n \ll 1$.

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

Fact: $E[X] = \lambda$.

Proof:

$$\begin{aligned} E[X] &= \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\ &= e^{-\lambda} \lambda e^{\lambda} = \lambda. \end{aligned}$$

□

Summary.

Distributions

- $U[1,\ldots,n] : Pr[X=m] = \frac{1}{n}, m = 1,\ldots,n;$
  $E[X] = \frac{n+1}{2};$
- $B(n,p) : Pr[X=m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0,\ldots,n;$
  $E[X] = np;$
- $G(p) : Pr[X=n] = (1-p)^{n-1} p, n = 1,2,\ldots;$
  $E[X] = \frac{1}{p};$
- $P(\lambda) : Pr[X=n] = \frac{\lambda^n}{n!} e^{-\lambda}, n \geq 0;$
  $E[X] = \lambda.$