# CS70: Lecture 26.

Recap of Distributions, Variance

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**Recap of Distributions, Variance**

1. Review: Distributions (Poisson)
2. Variance
3. Independence of Random Variables

# Review: Distributions

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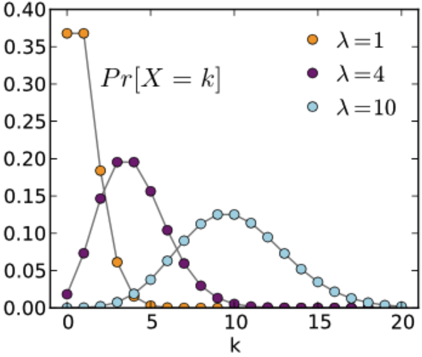

The Poisson distribution shows up in a lot of “real world” applications. Here is a partial list:

- the number of bankruptcies that are filed in a month
- the number of arrivals at a car wash in one hour
- the number of arrivals at the Cory & Hearst bus-stop
- the number of network failures per day
- the number of asthma patient arrivals in a given hour at a walk-in clinic
- the number of customer arrivals at McDonald’s per day
- the number of birth, deaths, marriages, divorces, suicides, and homicides over a given period of time
- the number of visitors to a web site per minute

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For (1) we used $m \ll n$; for (2) we used $(1 - a/n) \approx e^{-a/n}$ for $a/n \ll 1$.

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**Definition** Poisson Distribution with parameter $\lambda > 0$

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□

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**Siméon Poisson**

Siméon Denis Poisson (1781–1840)

# Equal Time: B. Geometric

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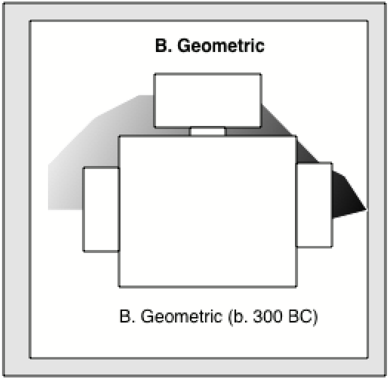

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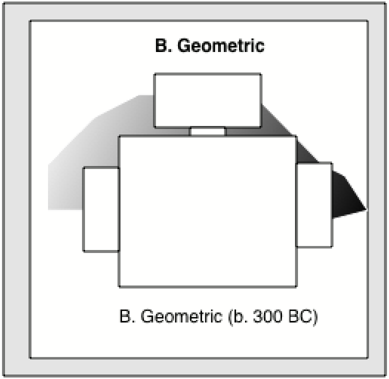


I could not find a picture of D. Binomial, sorry.

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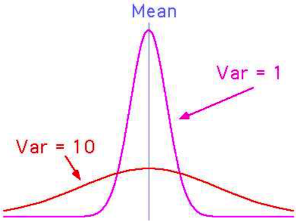

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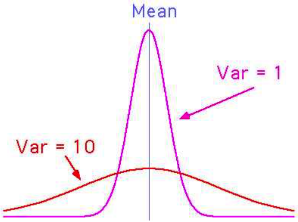


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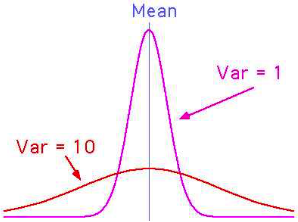


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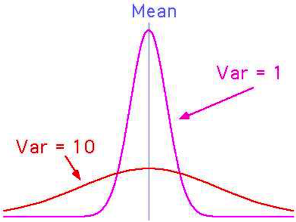


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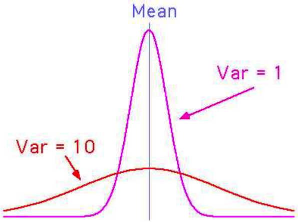


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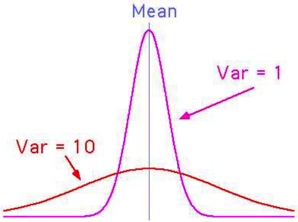


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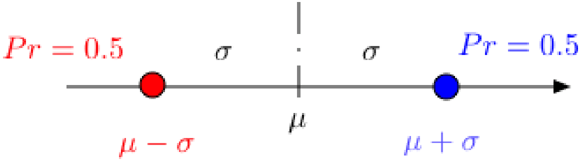

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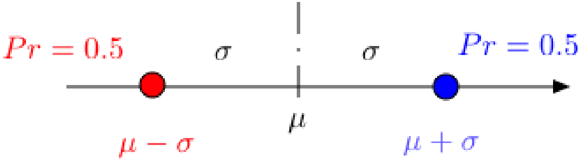


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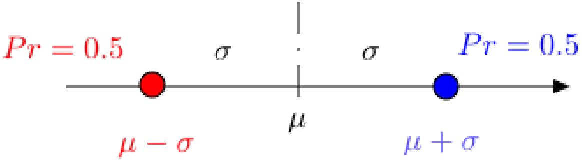


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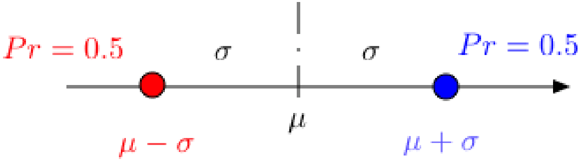


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Exercise: How big can you make $\frac{\sigma(X)}{E[|X-E[X]|]}$?

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This gives

$$var(X) = \frac{1 + 3n + 2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$

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$$\begin{aligned} var(X+Y) &= E((X+Y)^2) = E(X^2 + 2XY + Y^2) \\ &= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2) \\ &= var(X) + var(Y). \end{aligned}$$

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