# EECS 70: Lecture 27.

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3. Recap of indep. rand. variables: Variance of $B(n,p)$
4. Conditioning of Random Variables (revisit $G(p)$)

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**Joint Distribution:** $P[X=x, Y=y]$.
**Marginal Distributions:** $P[X=x]$ and $P[Y=y]$.
Important for inference.

# Two random variables, same outcome space.

Experiment: pick a random person.

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| $X$ | 0 | 1 | 2 | 3 | 5 | 40 | All |
|---|---|---|---|---|---|---|---|
| $P$ | 0.3 | 0.05 | 0.05 | 0.05 | 0.05 | 0.1 | 0.4 |

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| $Y$ | 0 | 1 | 5 | 10 |
|---|---|---|---|---|
| $P$ | 0.3 | 0.1 | 0.1 | 0.5 |

# Joint distribution: Example.

The **joint distribution** of *X* and *Y* is:

| Y/X | 0 | 1 | 2 | 3 | 5 | 40 | All | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.15 | 0 | 0 | 0 | 0 | 0.1 | 0.05 | =0.3 |
| 1 | 0 | 0.05 | 0.05 | 0 | 0 | 0 | 0 | =0.1 |
| 5 | 0 | 0 | 0 | 0.05 | 0.05 | 0 | 0 | =0.1 |
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| 1 | 0 | 0.05 | 0.05 | 0 | 0 | 0 | 0 | =0.1 |
| 5 | 0 | 0 | 0 | 0.05 | 0.05 | 0 | 0 | =0.1 |
| 10 | 0.15 | 0 | 0 | 0 | 0 | 0 | 0.35 | =0.5 |
| | =0.3 | =0.05 | =0.05 | =0.05 | =0.05 | =0.1 | =0.4 | |

Is this a valid distribution? Yes!

Notice that $P[X = a]$ and $P[Y = b]$ are (marginal) distributions!

# Joint distribution: Example.

The **joint distribution** of $X$ and $Y$ is:

| Y/X | 0 | 1 | 2 | 3 | 5 | 40 | All | |
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For example, if I tell you someone watched 5 episodes of Westworld, they definitely didn't watch all the episodes of GoT.

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Don't need a huge table of probabilities like the previous slide.

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□

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&= \frac{1 + 3n + 2n^2}{3} - \frac{(n+1)^2}{2}.
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$p_{X|Y}(x \mid y)$ is called the **conditional distribution or conditional probability mass function** (pmf) of $X$ given $Y$

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Let's visit the mean and variance of the geometric distribution using conditional expectation.

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Derive the variance for $X \sim G(p)$ by finding $E[X^2]$ using conditioning.

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- Conditional expectation: useful for mean & variance calculations