Continuous Probability
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1. Conditional Probability (Recap: revisit $G(p)$)
2. Continuous Probability: Examples
3. Continuous Probability: Events
4. Continuous Random Variables
Recap: Conditional distributions

$X \mid Y$ is a RV:

$$\sum_x p_{X \mid Y}(x \mid y) = \sum_x \frac{p_{XY}(x, y)}{p_Y(y)} = 1$$
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**Multiplication or Product Rule:**

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p_{XY}(x, y) = p_X(x)p_{Y \mid X}(y \mid x) = p_Y(y)p_{X \mid Y}(x \mid y)
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Nothing special about just two random variables, naturally extends to more.

Let's visit the mean and variance of the geometric distribution using conditional expectation.
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Total Probability Theorem: If \( A_1, A_2, \ldots, A_N \) partition \( \Omega \), and \( P[A_i] > 0 \ \forall \ i \), then

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p_X(x) = \sum_{i=1}^{N} P[A_i]P[X = x \mid A_i]
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Revisiting mean of geometric RV $X \sim G(p)$

$X$ is memoryless

$$P[X = n + m \mid X > n] = P[X = m].$$
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Why? (Recall \( E[g(X)] = \sum_l g(l)P[X = l] \))
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$$= \sum_{k=2}^{\infty} kP[X = k-1] \quad (memoryless)$$
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$$= E[X + 1]$$
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$$\Rightarrow E[X] = p \cdot 1 + (1 - p)(E[X] + 1)$$
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\[ \Rightarrow E[X] = p + 1 - p + E[X] - pE[X] \]
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Derive the variance for $X \sim G(p)$ by finding $E[X^2]$ using conditioning.
Summary of Conditional distribution

For Random Variables $X$ and $Y$, $P[X = x \mid Y = k]$ is the conditional distribution of $X$ given $Y = k$.
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Numerator: Joint distribution of $(X, Y)$. 

Aside: surprising result using conditioning of RVs:

**Theorem**: If $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$ are independent, then $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$. 

"Sum of independent Poissons is Poisson."
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Intuition based on Binomial limiting behavior
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Intuition based on Binomial limiting behavior

- $X_1 \sim B(n, p_1)$ where $p_1 = \frac{\lambda_1}{n}$, $n$ is large, $\lambda_1$ is constant

Question: What is $(a good approximation to)$ $Y = X_1 + X_2$?

($X_1, X_2$ independent)

$X_1$: $\quad \text{T T T T H T T T H}$

$X_2$: $\quad \text{T T H T T T T T H}$

$Y$: $\quad \text{T T H T H T T T}$

Intuition: If $p_1 = \frac{\lambda_1}{n}$ and $p_2 = \frac{\lambda_2}{n}$, then $p_1 p_2 = \frac{\lambda_1 \lambda_2}{n^2}$.

$\Rightarrow 2H$ will essentially NEVER appear!
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**Intuition** based on Binomial limiting behavior

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Question: What is (a good approximation to) $Y = X_1 + X_2$? ($X_1, X_2$ independent)
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\[
X_1 : \quad T \quad T \quad T \quad T \quad H \quad T \quad T \quad T \quad T \quad \cdots \quad H \quad \cdots
\]

\( H \) appears with probability \( p_1 \)
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$X_1: \quad T \quad T \quad T \quad T \quad H \quad T \quad T \quad T \quad T \quad \cdots \quad H \quad \cdots$

$H$ appears with probability $p_1$

$X_2: \quad T \quad T \quad H \quad T \quad T \quad T \quad T \quad T \quad T \quad \cdots \quad H \quad \cdots$
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$H$ appears with probability $p_1$

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$X_2: \quad T \quad T \quad H \quad T \quad T \quad T \quad T \quad T \quad T \quad \cdots \quad H \quad \cdots$

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$Y: \quad T \quad T \quad H \quad T \quad H \quad T \quad T \quad T \quad \cdots \quad 2H \quad \cdots$
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$H$ appears with probability $p_1 + p_2$, $2H$ appears with $p_1 p_2$
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$X_1 : \ T \ T \ T \ T \ T \ H \ T \ T \ T \ T \ \ldots \ H \ \ldots$

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$Y : \ T \ T \ H \ T \ H \ T \ T \ T \ T \ \ldots \ 2H \ \ldots$

$H$ appears with probability $p_1 + p_2$, $2H$ appears with $p_1 p_2$

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Question: What is (a good approximation to) $Y = X_1 + X_2$? ($X_1, X_2$ independent)

- $X_1: T \ T \ T \ T \ H \ T \ T \ T \ T \ T \ \cdots \ H \ \cdots$
  - $H$ appears with probability $p_1$

- $X_2: T \ T \ H \ T \ T \ T \ T \ T \ T \ T \ \cdots \ H \ \cdots$
  - $H$ appears with probability $p_2$

- $Y: T \ T \ H \ T \ H \ T \ T \ T \ T \ T \ \cdots \ 2H \ \cdots$
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$\Rightarrow 2H$ will essentially NEVER appear!
Sum of Independent Poissons is Poisson

Let’s define events:

- A: Every $Y_i$ has H or T for $i = 1, 2, \ldots, n$
- D: At least one $Y_i$ has $2H$ for $i = 1, 2, \ldots, n$
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We have $A$ and $D$ partition $\Omega$, so

$$P[Y = k] = P[Y = k \mid A]P[A] + P[Y = k \mid D]P[D]$$
### Sum of Independent Poissons is Poisson

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$$P[D] = P[\bigcup_{i=1}^{n} (Y_i \text{ is 2H})]$$
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$$P[D] = P[\bigcup_{i=1}^{n}(Y_i \text{ is 2H})]$$

$$\leq \sum_{i=1}^{n} P[Y_i \text{ is 2H}]$$
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$$P[Y = k] = P[Y = k \mid A]P[A] + P[Y = k \mid D]P[D]$$

$$P[D] = P[\bigcup_{i=1}^{n}(Y_i \text{ is } 2H)]$$
$$\leq \sum_{i=1}^{n} P[Y_i \text{ is } 2H]$$
$$\leq \sum_{i=1}^{n} \frac{\lambda_1 \lambda_2}{n^2}$$
Sum of Independent Poissons is Poisson

Let’s define events:

- A: Every $Y_i$ has H or T for $i = 1, 2, \cdots, n$
- D: At least one $Y_i$ has 2H for $i = 1, 2, \cdots, n$

We have $A$ and $D$ partition $\Omega$, so

$$P[Y = k] = P[Y = k \mid A]P[A] + P[Y = k \mid D]P[D]$$

$$P[D] = P[\bigcup_{i=1}^{n} (Y_i \text{ is } 2H)]$$

$$\leq \sum_{i=1}^{n} P[Y_i \text{ is } 2H]$$

$$\leq \sum_{i=1}^{n} \frac{\lambda_1 \lambda_2}{n^2} = \frac{\lambda_1 \lambda_2}{n}$$
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$P[Y = k \mid A] \sim B(n, p_1 + p_2)$
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Limit: "\( \text{Poisson}(\lambda_1) + \text{Poisson}(\lambda_2) = \text{Poisson}(\lambda_1 + \lambda_2) \)"
Continuous Probability: Why do we need it?

Many settings involve uncertainty in quantities like time, distance, velocity, temperature, etc. that are *continuous-valued*. 

*You break a stick at two points chosen independently uniformly at random.* 

What is the probability you can make a triangle with the three pieces? 

*In digital video and audio, one represents a continuous value by a finite number of bits.* 

This introduces an error perceived as noise: the *quantization noise*. 

*What is the power of that noise?*
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Continuous Probability: Uniformly at Random in \([0, 1]\).

Choose a real number \(X\), uniformly at random in \([0, 1]\). What is the probability that \(X\) is exactly equal to \(\frac{1}{3}\)? Well, \(0\). What is the probability that \(X\) is exactly equal to \(0.6\)? Again, \(0\). In fact, for any \(x\in [0, 1]\), one has \(\Pr[X = x] = 0\).

How should we then describe ‘choosing uniformly at random in \([0, 1]\)’? Here is the way to do it: \(\Pr[X \in [a, b]] = b - a, \forall 0 \leq a \leq b \leq 1\). Makes sense: \(b - a\) is the fraction of \([0, 1]\) that \([a, b]\) covers.
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\text{Pr}([a, b]) = \frac{\text{length of } [a, b]}{\text{length of } [0, 1]} = \frac{b-a}{1}.
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Intervals like $[a, b] \subseteq \Omega = [0, 1]$ are events. More generally, events in this space are unions of intervals. 

Example: the event $A$ - "within 0.2 of 0 or 1" is 

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A = [0, 0.2] \cup [0.8, 1].
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Thus, 

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\text{Pr}(A) = \text{Pr}([0, 0.2]) + \text{Pr}([0.8, 1]) = 0.4.
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More generally, if $A_n$ are pairwise disjoint intervals in $[0, 1]$, then 

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\text{Pr}(\bigcup A_n) := \sum \text{Pr}(A_n).
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Many subsets of $[0, 1]$ are of this form. Thus, the probability of those sets is well defined. We call such sets events.
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Let $[a, b]$ denote the **event** that the point $X$ is in the interval $[a, b]$. 

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Note: A radical change in approach. For a finite probability space, $\Omega = \{1, 2, \ldots, N\}$, we started with $\Pr[\omega] = p_\omega$. We then defined $\Pr[A] = \sum_{\omega \in A} p_\omega$ for $A \subset \Omega$.

We used the same approach for countable $\Omega$. For a continuous space, e.g., $\Omega = [0, 1]$, we cannot start with $\Pr[\omega]$, because this will typically be 0. Instead, we start with $\Pr[A]$ for some events $A$. Here, we started with $A = \text{interval, or union of intervals.}$
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![Diagram](image.png)
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Let us define $F(x) = \Pr[X \leq x]$. Then we have \\[ \Pr[X \in (a, b)] = \Pr[X \leq b] - \Pr[X \leq a] = F(b) - F(a). \] 

Thus, $F(\cdot)$ specifies the probability of all the events!
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![Probability Distribution](image-url)
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![Graph showing the cumulative distribution function](image)
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Thus, the probability of an event is the integral of \( f(x) \) over the event:

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Pr[X \in A] = \int_A f(x) \, dx.
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Think of $f(x)$ as describing how one unit of probability is spread over $[0, 1]$:

- This makes the probability automatically additive.
- We need $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x) \, dx = 1$.

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Discrete Approximation:

Fix $N \gg 1$ and let $\epsilon = \frac{1}{N}$.

Define $Y = n\epsilon$ if $(n-1)\epsilon < X \leq n\epsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \epsilon$ and $Y$ is discrete: $Y \in \{\epsilon, 2\epsilon, \ldots, N\epsilon\}$.

Also, $\Pr[Y = n\epsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Thus, $X$ is 'almost discrete.'
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Nonuniformly at Random in $[0, 1]$. 

This figure shows a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x) \, dx = 1$. It defines another way of choosing $X$ at random in $[0, 1]$. Note that $X$ is more likely to be closer to 1 than to 0. One has $\Pr[X \leq x] = \int_{-\infty}^{x} f(u) \, du = x^2$ for $x \in [0, 1]$. Also, $\Pr[X \in (x, x+\varepsilon)] = \int_{x}^{x+\varepsilon} f(u) \, du \approx f(x) \varepsilon$. 
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For instance, $\Pr[X \in [0,1/3)] = \int_{0}^{1/3} x \, dx = \frac{2}{9}$.

Thus, $\Pr[X \in [0,1/3)] = \Pr[X \in [2/3,1)] = \frac{2}{9}$ and $\Pr[X \in [1/3,2/3)] = \frac{5}{9}$. 

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General Random Choice in $\mathbb{R}$

Let $F(x)$ be a nondecreasing function with $F(-\infty) = 0$ and $F(\infty) = 1$.

Define $X$ by $\Pr[X \in (a, b)] = F(b) - F(a)$ for $a < b$.

Also, for $a_1 < b_1 < a_2 < b_2 < \cdots < b_n$,

$$\Pr[X \in (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n)] = \Pr[X \in (a_1, b_1)] + \cdots + \Pr[X \in (a_n, b_n)] = F(b_1) - F(a_1) + \cdots + F(b_n) - F(a_n).$$

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Here, $F(x)$ is called the cumulative distribution function (cdf) of $X$ and $f(x)$ is the probability density function (pdf) of $X$.

To indicate that $F$ and $f$ correspond to the RV $X$, we will write them $F_X(x)$ and $f_X(x)$. 
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\Pr[X \in (a_1, b_1] \cup (a_2, b_2] \cup (a_n, b_n)] \\
= \Pr[X \in (a_1, b_1)] + \cdots + \Pr[X \in (a_n, b_n)] \\
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Let \( f(x) = \frac{d}{dx} F(x) \).
General Random Choice in \( \mathbb{R} \)

Let \( F(x) \) be a nondecreasing function with \( F(-\infty) = 0 \) and \( F(+\infty) = 1 \).

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Here, \( F(x) \) is called the cumulative distribution function (cdf) of \( X \).
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To indicate that $F$ and $f$ correspond to the RV $X$, we will write $F_X(x)$ and $f_X(x)$. 
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\[ Pr[X \in (x, x + \varepsilon)] \]

Thus, the pdf is the 'local probability by unit length.' It is the 'probability density.'
$Pr[X \in (x, x + \varepsilon)]$

An illustration of $Pr[X \in (x, x + \varepsilon)] \approx f_X(x)\varepsilon$:
\( Pr[X \in (x, x + \varepsilon)] \)

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An illustration of $Pr[X \in (x, x + \varepsilon)] \approx f_X(x)\varepsilon$:

Thus, the pdf is the ‘local probability by unit length.’

It is the ‘probability density.’
Discrete Approximation

Fix $\varepsilon \ll 1$ and let $Y = n \varepsilon$ if $X \in (n \varepsilon, (n + 1) \varepsilon]$. Thus, $\Pr[Y = n \varepsilon] = \mathcal{F}_X((n + 1) \varepsilon) - \mathcal{F}_X(n \varepsilon)$.

Note that $|X - Y| \leq \varepsilon$ and $Y$ is a discrete random variable. Also, if $f_X(x) = \frac{d}{dx} \mathcal{F}_X(x)$, then $\mathcal{F}_X(x + \varepsilon) - \mathcal{F}_X(x) \approx f_X(x) \varepsilon$.

Hence, $\Pr[Y = n \varepsilon] \approx f_X(n \varepsilon) \varepsilon$.

Thus, we can think of $X$ of being almost discrete with $\Pr[X = n \varepsilon] \approx f_X(n \varepsilon) \varepsilon$. 
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Fix $\epsilon \ll 1$ and let $Y = n\epsilon$ if $X \in (n\epsilon, (n+1)\epsilon]$. Thus, $Pr[Y = n\epsilon] = F_X((n+1)\epsilon) - F_X(n\epsilon)$. 

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Example: CDF

Example: hitting random location on gas tank.

Random location on circle.

Random Variable: \( Y \)

distance from center.

Probability within \( y \) of center:

\[
\Pr[Y \leq y] = \frac{\text{area of small circle}}{\text{area of dartboard}} = \frac{\pi y^2}{\pi} = y^2.
\]

Hence,

\[
F_Y(y) = \Pr[Y \leq y] = \begin{cases} 
0 & \text{for } y < 0 \\
y^2 & \text{for } 0 \leq y \leq 1 \\
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\end{cases}
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Example: CDF

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Distance from center.
Probability within $y$ of center:
$P_r\left[Y \leq y\right] = \frac{\text{area of small circle}}{\text{area of dartboard}} = \frac{\pi y^2}{\pi} = y^2$.

Hence, $F_Y(y) = P_r\left[Y \leq y\right] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$.
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Calculation of event with dartboard.

Probability between .5 and .6 of center?

Recall CDF $F_Y(y) = \begin{cases} 
0 & \text{for } y < 0 \\
y^2 & \text{for } 0 \leq y \leq 1 \\
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\end{cases}$

$$Pr[0.5 < Y \leq 0.6] = F_Y(0.6) - F_Y(0.5) = 0.36 - 0.25 = 0.11$$
Calculation of event with dartboard..

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\[ \Pr[0.5 < Y \leq 0.6] = \Pr[Y \leq 0.6] - \Pr[Y \leq 0.5] = F_Y(0.6) - F_Y(0.5) \]

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\[
f_Y(y) = F'_Y(y) = \begin{cases} 
0 & \text{for } y < 0 \\
2y & \text{for } 0 \leq y \leq 1 \\
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The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.
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\end{cases}
\]

\[
f_Y(y) = F_Y'(y) = \begin{cases} 
0 & \text{for } y < 0 \\
2y & \text{for } 0 \leq y \leq 1 \\
0 & \text{for } y > 1
\end{cases}
\]

The cumulative distribution function (cdf) and probability distribution function (pdf) give full information. Use whichever is convenient.
Target
Target

Random Variable

Event \( \{Y \leq y\} \)

Outcome

\[ F_Y(y) \]

\[ f_Y(y) \]

\[ y^2 \]

\[ 2y \]
$U[a, b]$
$U[a, b]$
The exponential distribution with parameter $\lambda > 0$ is defined by

$$f_X(x) = \lambda e^{-\lambda x} \{ x \geq 0 \}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$

Note that $\Pr[X > t] = e^{-\lambda t}$ for $t > 0$. 
The exponential distribution with parameter $\lambda > 0$ is defined by

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**Expo(λ)**

The exponential distribution with parameter $λ > 0$ is defined by

$$f_X(x) = λ e^{-λx} 1\{x ≥ 0\}$$

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\end{cases}$$

Note that $Pr[X > t] = e^{-λt}$ for $t > 0$. 
Random Variables

Continuous random variable $X$, specified by

1. $F_X(x) = \Pr[X \leq x]$ for all $x$. 

Recall that $\Pr[X \in (x, x+\delta)] \approx f_X(x)\delta$. 

Think of $X$ taking discrete values $n\delta$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ with $\Pr[X = n\delta] = f_X(n\delta)\delta$. 

Random Variables

Continuous random variable $X$, specified by

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**Cumulative Distribution Function (cdf).**
Random Variables

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$Pr[a < X \leq b] = F_X(b) - F_X(a)$
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1.1 $0 \leq F_X(x) \leq 1$ for all $x \in \mathbb{R}$. 


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**Probability Density Function (pdf).**
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2. Or $f_X(x)$, where $F_X(x) = \int_{-\infty}^{x} f_X(u)du$ or $f_X(x) = \frac{d(F_X(x))}{dx}$.

Probability Density Function (pdf).

$Pr[a < X \leq b] = \int_{a}^{b} f_X(x)dx = F_X(b) - F_X(a)$
Random Variables

Continuous random variable $X$, specified by

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Recall that $Pr[X \in (x, x+\delta)] \approx f_X(x) \delta$.
Think of $X$ taking discrete values $n\delta$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ with $Pr[X = n\delta] = f_X(n\delta)\delta$. 

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Summary

Continuous Probability

1. pdf:

\[
Pr\left[X \in (x, x + \delta]\right] = f_X(x)\delta.
\]

2. CDF:

\[
Pr\left[X \leq x\right] = F_X(x) = \int_{-\infty}^{\infty} f_X(y) dy.
\]

3. Uniform [a, b]:

\[
f_X(x) = \begin{cases} 
\frac{1}{b-a} & \text{if } a \leq x \leq b \\
0 & \text{otherwise} 
\end{cases};
\]

\[
F_X(x) = \begin{cases} 
\frac{x-a}{b-a} & \text{if } a \leq x \leq b \\
1 & \text{if } x > b \\
0 & \text{if } x < a 
\end{cases}.
\]

4. Exponential (\lambda):

\[
f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0;
\]

\[
F_X(x) = 1 - e^{-\lambda x} \quad \text{for } x \leq 0.
\]

5. Target:

\[
f_X(x) = 2x \quad \text{if } 0 \leq x \leq 1;
\]

\[
F_X(x) = x^2 \quad \text{for } 0 \leq x \leq 1.
\]
Summary

Continuous Probability

1. pdf: $Pr[X \in (x, x + \delta)] = f_X(x)\delta$. 
Summary

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1. pdf: \( Pr[X \in (x, x + \delta)] = f_X(x)\delta. \)
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4. $Expo(\lambda)$:
1. pdf: \( Pr[X \in (x, x + \delta)] = f_X(x)\delta. \)

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3. \( U[a, b] \): \( f_X(x) = \frac{1}{b-a} \{ a \leq x \leq b \}; F_X(x) = \frac{x-a}{b-a} \) for \( a \leq x \leq b. \)

4. \( Expo(\lambda) \):
   \( f_X(x) = \lambda \exp\{-\lambda x\} \{ x \geq 0 \}; \)
Continuous Probability

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Continuous Probability

1. **pdf:** \( Pr[X \in (x, x + \delta)] = f_X(x)\delta \).
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5. **Target:** \( f_X(x) = 2x 1\{0 \leq x \leq 1\}; \)
Summary

Continuous Probability

1. **pdf**: $Pr[X \in (x, x + \delta)] = f_X(x) \delta$.
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