CS70: Lecture 29

Continuous Probability (continued)

1. Review: CDF, PDF
2. Examples
3. Properties
4. Expectation of continuous random variables

Review: Continuous Probability

Key idea: For a continuous RV, $Pr[X = x] = 0$ for all $x \in \Re$.

Examples: Uniform in $[0,1]$;

Thus, one cannot define $Pr[\text{outcome}]$, then $Pr[\text{event}]$.

Instead, one starts by defining $Pr[\text{event}]$.

Thus, one defines $Pr[X \in (-\infty, x]] = Pr[X \leq x] =: F_X(x), x \in \Re$.

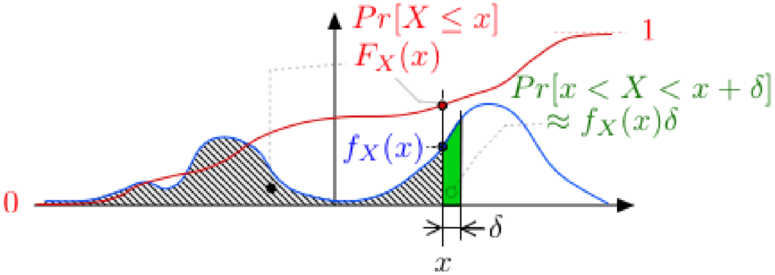
Then, one defines $f_X(x) := \frac{d}{dx} F_X(x)$.

Hence, $f_X(x)\varepsilon \approx Pr[X \in (x, x+\varepsilon)]$.

$F_X(\cdot)$ is the cumulative distribution function (CDF) of X.

$f_X(\cdot)$ is the probability density function (PDF) of X.

A Picture



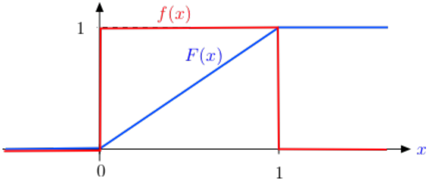
The pdf $f_X(x)$ is a nonnegative function that integrates to 1.

The cdf $F_X(x)$ is the integral of f_X.

$$Pr[x < X < x + \delta] \approx f_X(x)\delta$$

$$Pr[X \leq x] = F_x(x) = \int_{-\infty}^{x} f_X(u)du$$

Uniformly at Random in $[0,1]$.



$$Pr[X \in (a,b]] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a).$$

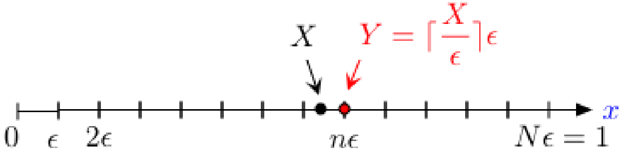
An alternative view is to define $f(x) = \frac{d}{dx}F(x) = 1\{x \in [0,1]\}$. Then

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Thus, the probability of an event is the integral of $f(x)$ over the event:

$$Pr[X \in A] = \int_A f(x)dx.$$

Uniformly at Random in $[0,1]$.



Discrete Approximation: Fix $N \gg 1$ and let $\varepsilon = 1/N$.

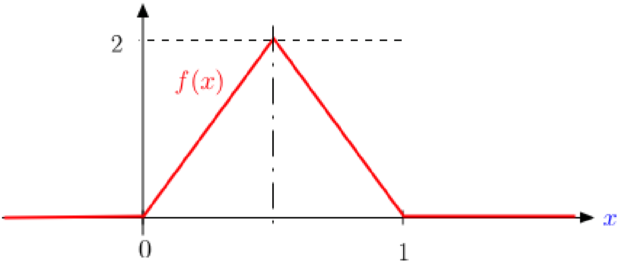
Define $Y = n\varepsilon$ if $(n-1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \varepsilon$ and Y is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Thus, X is 'almost discrete.'

Nonuniform Choice at Random in $[0,1]$.



This figure shows yet a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$.

It defines another way of choosing X at random in $[0,1]$.

Note that X is more likely to be closer to $1/2$ than to 0 or 1.

For instance, $Pr[X \in [0,1/3]] = \int_0^{1/3} 4xdx = 2\left[x^2\right]_0^{1/3} = \frac{2}{9}$.

Thus, $Pr[X \in [0,1/3]] = Pr[X \in [2/3,1]] = \frac{2}{9}$ and $Pr[X \in [1/3,2/3]] = \frac{5}{9}$.

General Random Choice in $\Re$

Let $F(x)$ be a nondecreasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$.

Define X by $Pr[X \in (a, b]] = F(b) - F(a)$ for $a < b$. Also, for $a_1 < b_1 < a_2 < b_2 < \cdots < b_n$,

$$
\begin{aligned}
Pr[X \in (a_1, b_1] \cup (a_2, b_2] \cup (a_n, b_n]] \\
&= Pr[X \in (a_1, b_1]] + \cdots + Pr[X \in (a_n, b_n]] \\
&= F(b_1) - F(a_1) + \cdots + F(b_n) - F(a_n).
\end{aligned}
$$

Let $f(x) = \frac{d}{dx} F(x)$. Then,

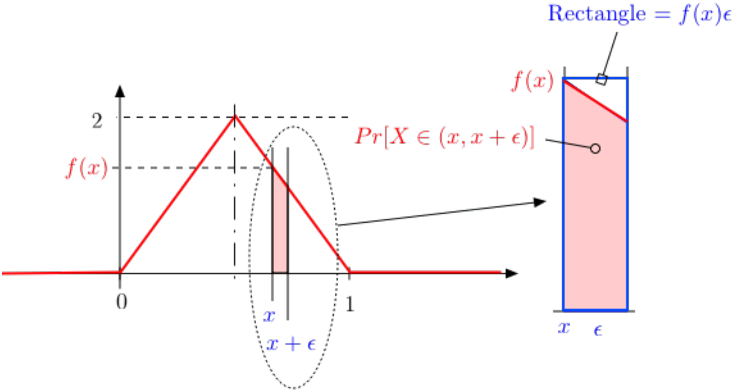
$$Pr[X \in (x, x + \varepsilon]] = F(x + \varepsilon) - F(x) \approx f(x)\varepsilon.$$

Here, $F(x)$ is called the cumulative distribution function (cdf) of X and $f(x)$ is the probability density function (pdf) of X.

To indicate that F and f correspond to the RV X, we will write them $F_X(x)$ and $f_X(x)$.

$Pr[X \in (x, x+\varepsilon)]$

An illustration of $Pr[X \in (x, x+\varepsilon)] \approx f_X(x)\varepsilon$:



Thus, the pdf is the 'local probability by unit length.'

It is the 'probability density.'

Discrete Approximation

Fix $\varepsilon \ll 1$ and let $Y = n\varepsilon$ if $X \in (n\varepsilon, (n+1)\varepsilon]$.

Thus, $Pr[Y = n\varepsilon] = F_X((n+1)\varepsilon) - F_X(n\varepsilon)$.

Note that $|X - Y| \leq \varepsilon$ and Y is a discrete random variable.

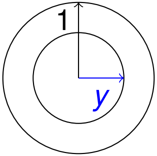
Also, if $f_X(x) = \frac{d}{dx}F_X(x)$, then $F_X(x+\varepsilon) - F_X(x) \approx f_X(x)\varepsilon$.

Hence, $Pr[Y = n\varepsilon] \approx f_X(n\varepsilon)\varepsilon$.

Thus, we can think of X of being almost discrete with $Pr[X = n\varepsilon] \approx f_X(n\varepsilon)\varepsilon$.

Example: CDF

Example: hitting random location on gas tank.
Random location on circle.



Random Variable: Y distance from center.
Probability within y of center:

$$\begin{aligned} Pr[Y \le y] &= \frac{\text{area of small circle}}{\text{area of dartboard}} \\ &= \frac{\pi y^2}{\pi} = y^2. \end{aligned}$$

Hence,

$$F_Y(y) = Pr[Y \le y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Calculation of event with dartboard..

Probability between .5 and .6 of center?
Recall CDF.

$$F_Y(y) = Pr[Y \le y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

$$\begin{aligned} Pr[0.5 < Y \le 0.6] &= Pr[Y \le 0.6] - Pr[Y \le 0.5] \\ &= F_Y(0.6) - F_Y(0.5) \\ &= .36 - .25 \\ &= .11 \end{aligned}$$

PDF.

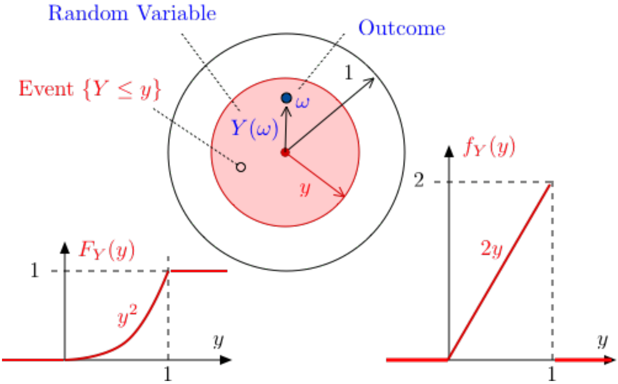
Example: “Dart” board.
Recall that

$$F_Y(y) = Pr[Y \leq y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

$$f_Y(y) = F'_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{for } y > 1 \end{cases}$$

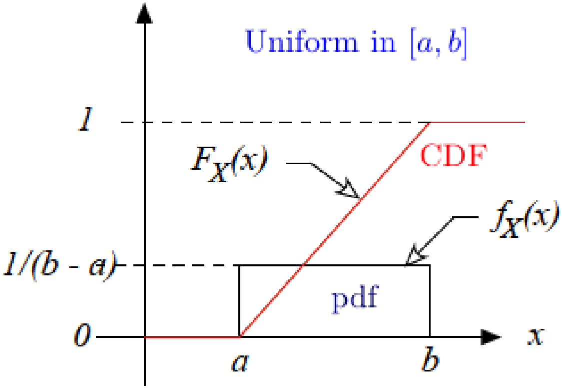
The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.
Use whichever is convenient.

Target



Random Variable
Outcome
Event $\{Y \leq y\}$
1
ω
$Y(\omega)$
y
$F_Y(y)$
1
y^2
y
1
$f_Y(y)$
2
$2y$
y
1

$U[a, b]$

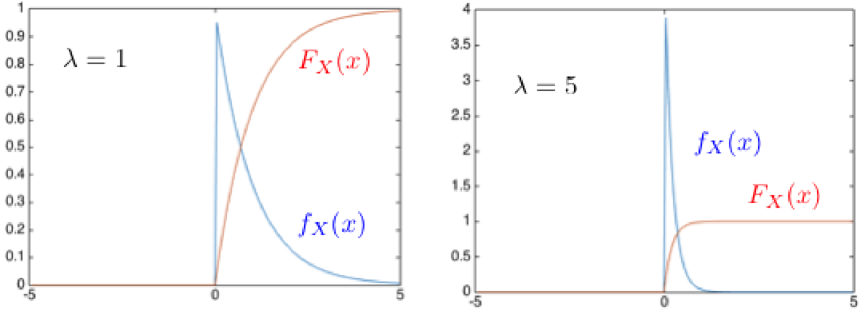


$Expo(\lambda)$

The exponential distribution with parameter $\lambda > 0$ is defined by

$$f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$



Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

Some Properties

1. ***Expo* is memoryless.** Let $X = Expo(\lambda)$. Then, for $s, t > 0$,

$$\begin{aligned} Pr[X > t + s \mid X > s] &= \frac{Pr[X > t + s]}{Pr[X > s]} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} \\ &= Pr[X > t]. \end{aligned}$$

'Used is a good as new.'

2. Scaling *Expo*. Let $X = Expo(\lambda)$ and $Y = aX$ for some $a > 0$. Then

$$\begin{aligned} Pr[Y > t] &= Pr[aX > t] = Pr[X > t/a] \\ &= e^{-\lambda(t/a)} = e^{-(\lambda/a)t} = Pr[Z > t] \text{ for } Z = Expo(\lambda/a). \end{aligned}$$

Thus, $a \times Expo(\lambda) = Expo(\lambda/a)$.

Also, $Expo(\lambda) = \frac{1}{\lambda} Expo(1)$.

Expectation

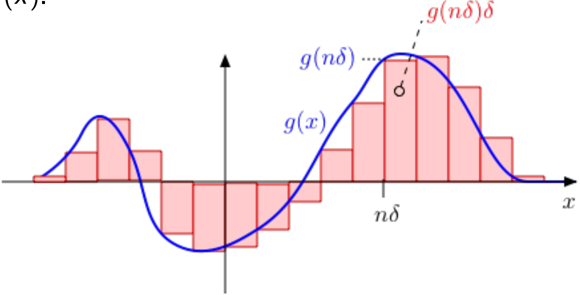
Definition: The **expectation** of a random variable X with pdf $f(x)$ is *defined* as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Justification: Say $X = n\delta$ w.p. $f_X(n\delta)\delta$ for $n \in \mathbb{Z}$. Then,

$$E[X] = \sum_n (n\delta) Pr[X = n\delta] = \sum_n (n\delta) f_X(n\delta)\delta = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Indeed, for any g, one has $\int g(x)dx \approx \sum_n g(n\delta)\delta$. Choose $g(x) = x f_X(x)$.



Examples of Expectation

1. $X = U[0,1]$. Then, $f_X(x) = 1\{0 \le x \le 1\}$. Thus,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x.1 dx = [\frac{x^2}{2}]_0^1 = \frac{1}{2}.$$

2. $X =$ distance to 0 of dart shot uniformly in unit circle. Then $f_X(x) = 2x1\{0 \le x \le 1\}$. Thus,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x.2x dx = [\frac{2x^3}{3}]_0^1 = \frac{2}{3}.$$

Examples of Expectation

3. $X = Expo(\lambda)$. Then, $f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$. Thus,

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} dx = -\int_0^\infty x de^{-\lambda x}.$$

Recall the integration by parts formula:

$$\begin{aligned}
\int_a^b u(x) dv(x) &= \left[u(x)v(x)\right]_a^b - \int_a^b v(x) du(x) \\
&= u(b)v(b) - u(a)v(a) - \int_a^b v(x) du(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\infty x de^{-\lambda x} &= [xe^{-\lambda x}]_0^\infty - \int_0^\infty e^{-\lambda x} dx \\
&= 0 - 0 + \frac{1}{\lambda} \int_0^\infty de^{-\lambda x} = -\frac{1}{\lambda}.
\end{aligned}$$

Hence, $E[X] = \frac{1}{\lambda}$.

Linearity of Expectation

Theorem Expectation is linear.

Proof: 'As in the discrete case.' $\square$

Example 1: $X = U[a,b]$. Then

(a) $f_X(x) = \frac{1}{b-a} 1\{a \le x \le b\}$. Thus,

$$E[X] = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{a+b}{2}.$$

(b) $X = a + (b-a)Y$, $Y = U[0,1]$. Hence,

$$E[X] = a + (b-a)E[Y] = a + \frac{b-a}{2} = \frac{a+b}{2}.$$

Example 2: X, Y are $U[0,1]$. Then

$$E[3X - 2Y + 5] = 3E[X] - 2E[Y] + 5 = 3\frac{1}{2} - 2\frac{1}{2} + 5 = 5.5.$$

Summary

$\boxed{\text{Continuous Probability}}$

1. pdf: $Pr[X \in (x, x+\delta]] = f_X(x)\delta$.
2. CDF: $Pr[X \leq x] = F_X(x) = \int_{-\infty}^{x} f_X(y)dy$.
3. $U[a,b]$, $Expo(\lambda)$, target.
4. Expectation: $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$.
5. Expectation is linear.