Happy Monday!
Today:
Finish Note 2.
Begin Induction.

Proof by cases.

Theorem: $x^5 - x + 1 = 0$ has no solution in the rationals.
Proof:
First a lemma...
Lemma: If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.
Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma
$\Rightarrow$ no rational solution.

Proof of lemma: Assume a solution of the form $a/b$.

$$\left(\frac{a}{b}\right)^5 \frac{a}{b} - 1 = 0$$
Multiply by $b^5$,

$$a^5 - ab^4 + b^5 = 0$$

Case 1: $a$ odd, $b$ odd: odd - odd + odd = even. Not possible.
Case 2: $a$ even, $b$ odd: even - even + odd = even. Not possible.
Case 3: $a$ odd, $b$ even: odd - even + even = even. Not possible.
Case 4: $a$ even, $b$ even: even - even + even = even. Possible.
The fourth case is the only one possible, so the lemma follows.

Proof by cases.

Theorem: There exist irrational $x$ and $y$ such that $xy$ is rational.
Proof: First a lemma...
Lemma: If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.
Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma
$\Rightarrow$ no rational solution.

Proof of lemma: Assume a solution of the form $a/b$.

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Case 3: $a$ odd, $b$ even: odd - even + even = even. Not possible.
Case 4: $a$ even, $b$ even: even - even + even = even. Possible.
The fourth case is the only one possible, so the lemma follows.
Be really careful!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

$(x^2 - xy) = x^2 - y^2$

$x(x - y) = (x + y)(x - y)$

$x = x + y$

$1 = 2$

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$.

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Summary: Note 2.

**Direct Proof:**

To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

**By Contraposition:**

To Prove: $P \implies \neg Q$. Assume $\neg Q$. ... And we get...

$P(0)$ worked.

Careful when proving!

Don’t assume the theorem. Divide by zero. Watch converse. ...

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**Principle of Induction.**

$P(0) \land (\forall n \in \mathbb{N})P(n) \implies P(n + 1)$

And we get...

$(\forall n \in \mathbb{N})P(n)$.

...Yes for 0, and we can conclude Yes for 1...

and we can conclude Yes for 2......

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**Four Color Theorem.**

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.

**Proof:** By induction.

Base Case: $P(0)$ is $\emptyset^0 = 0$ is divisible by 3. $3 \mid (0^3 - 0)$

Induction Step: $(\forall k \in \mathbb{N}), P(k) \implies P(k + 1)$

Induction Hypothesis: $k^3 - k$ is divisible by 3.

$\left(k + 1\right)^3 - \left(k + 1\right) = k^3 + 3k^2 + 3k - (k + 1)$

By distributing and adding, we get:

$k^3 + 3k^2 + 3k = 3q + 3\left(k^2 + k\right)$

From our induction hypothesis, $3q + 3\left(k^2 + k\right)$ is divisible by 3.

Therefore, $\left(k + 1\right)^3 - \left(k + 1\right)$ is divisible by 3.

Thus, $(\forall k \in \mathbb{N})P(k) \implies P(k + 1)$

Thus, theorem holds by induction.

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Check Out: “Four corners”.

States connected at a point, can have same color. (Couldn’t find a map where they did though.)

Quick Test: Which states? Utah, Colorado, New Mexico, Arizona.
Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

Fact: Swapping red and blue gives another valid color.

Two color theorem: proof illustration.

Base Case.
1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.
   (Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k + 1)$.

Review Argument: $P(k) \implies P(k + 1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.
For any “edge”.
Ok before switch. Still ok, by “fact”.
Not ok before switch, must be on new line.
Switch changes one side,
So now two sides have different colors.

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.
Induction:
First Step (Base case).
Can step up the ladder of naturals. (Induction Step.)
Get to be on step $k$. (Use induction hypothesis.)
See you on Wednesday!