CS 70: Discrete Math and Probability

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Today:
  Finish Note 2.
  Begin Induction.

Proof by contradiction: example

Theorem: There are infinitely many primes.

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- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
- There is a prime *in between* 13 and $q = 30031$ that divides q.
- Proof assumed no primes *in between* p_k and q.

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Lemma: If x is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in Z$, *then both a and b are even.*

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The fourth case is the only one possible, so the lemma follows. $\square$

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Thus, we have irrational x and y with a rational x^y (i.e., 2).

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Question: Which case holds?

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$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}*\sqrt{2}} = \sqrt{2}^2 = 2.$$

Thus, we have irrational x and y with a rational x^y (i.e., 2).

One of the cases is true so theorem holds. $\square$

Question: Which case holds? Don't know!!!

Be careful.

Theorem: $3 = 4$

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Proof: Assume $3 = 4$.

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Start with $12 = 12$.

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Divide one side by 3 and the other by 4 to get

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Divide one side by 3 and the other by 4 to get
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By commutativity theorem holds. $\square$

Don't assume what you want to prove!

Be really careful!

Theorem: $1 = 2$
Proof:

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Proof: For $x = y$, we have

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Theorem: $1 = 2$
Proof: For $x = y$, we have

$$\begin{aligned} (x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y) \end{aligned}$$

Be really careful!

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Proof: For $x = y$, we have

$$\begin{aligned} (x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y) \\ x &= (x+y) \end{aligned}$$

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Theorem: $1 = 2$
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$$\begin{aligned}(x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y) \\ x &= (x+y) \\ x &= 2x\end{aligned}$$

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$P \implies Q$ does not mean $Q \implies P$.

Summary: Note 2.

Direct Proof:

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To Prove: $P \implies Q$.

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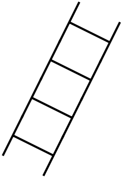

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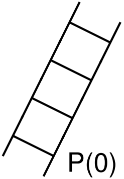

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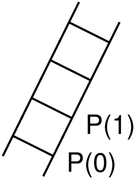

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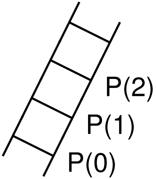

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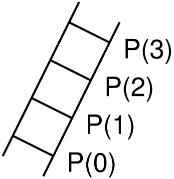

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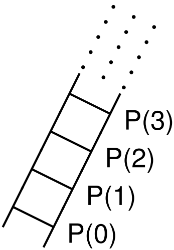

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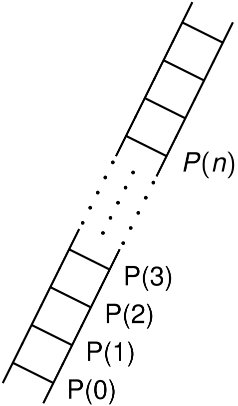

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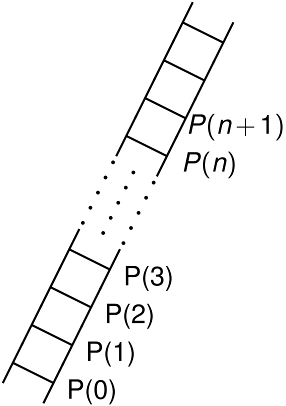

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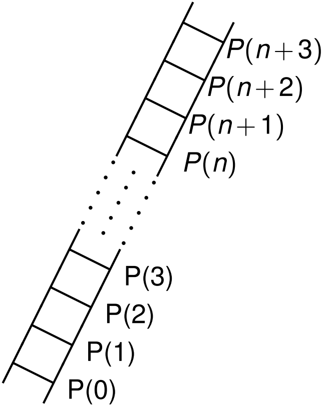

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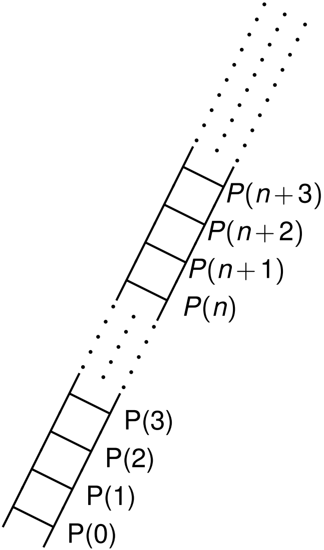

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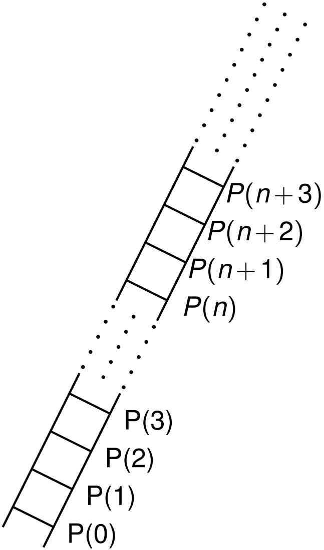

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Your favorite example of forever..

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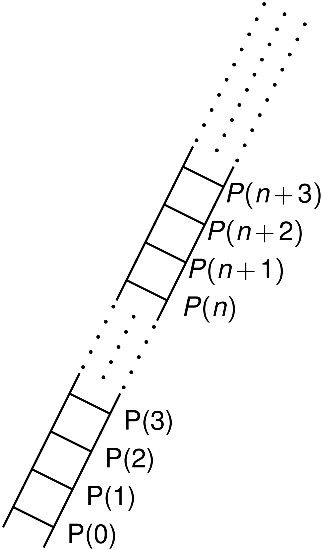

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Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n))$.

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 or $k^3 - k = 3q$ for some integer q.

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$(k+1)^3 - (k+1)$

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$(q + k^2 + k)$ is integer (closed under addition and multiplication).

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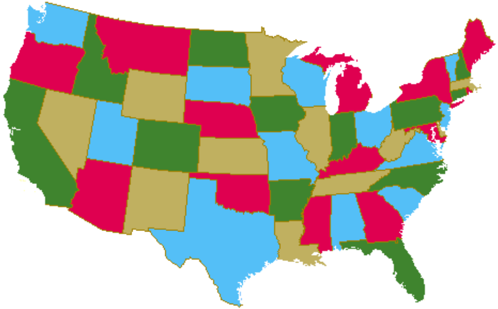
Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



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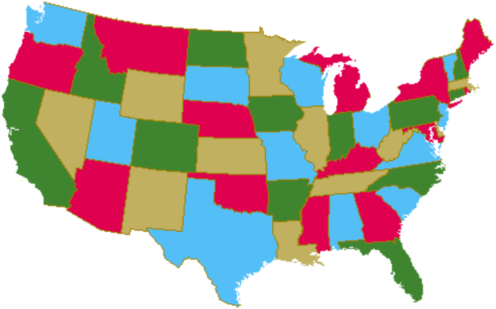

Check Out: “Four corners”.

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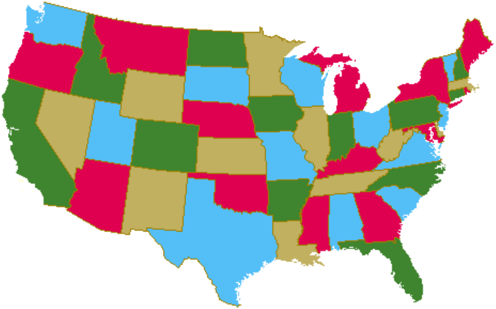

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States connected at a point, can have same color.

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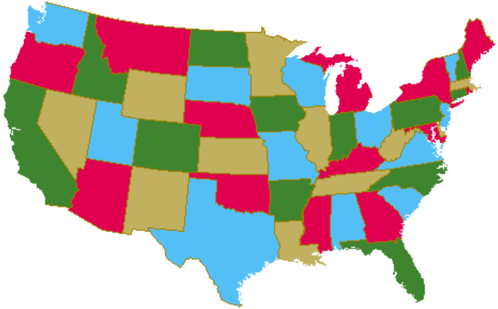

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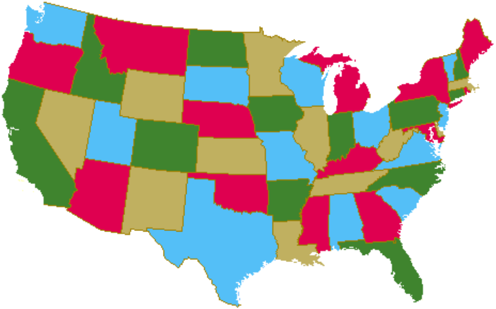

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Quick Test: Which states?

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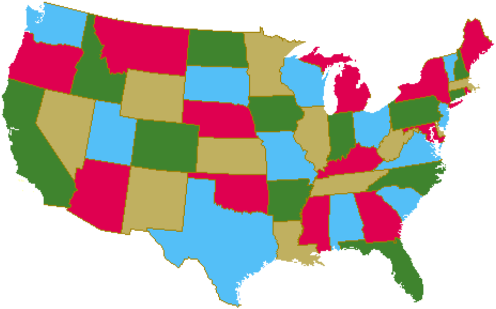

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Quick Test: Which states? Utah.

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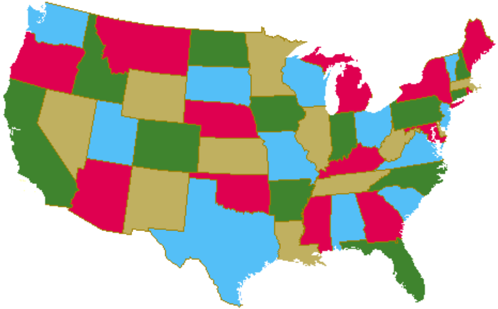

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Quick Test: Which states? Utah. Colorado.

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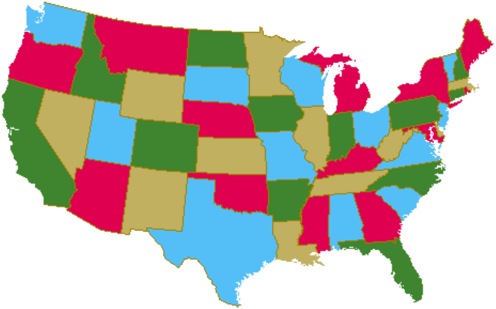

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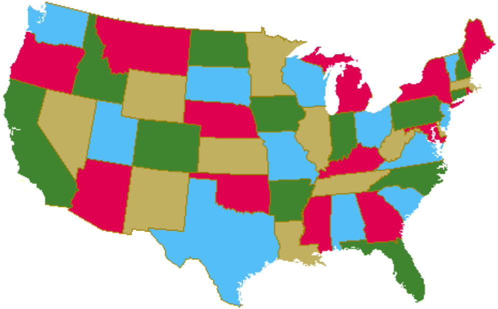

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Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

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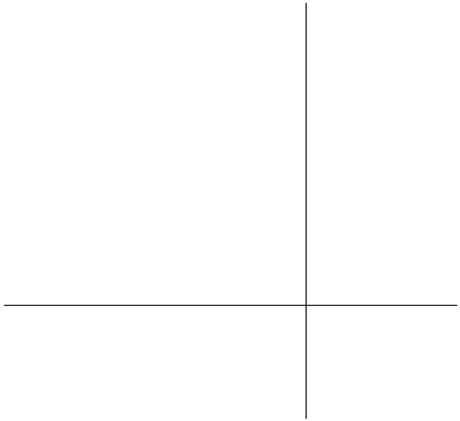

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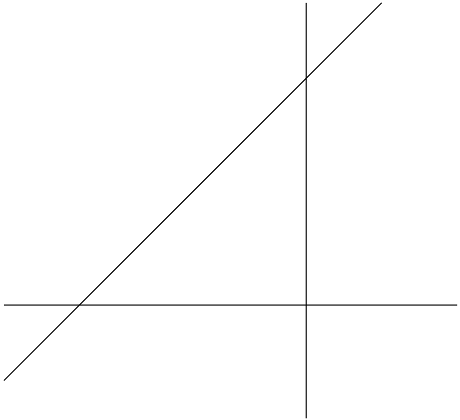

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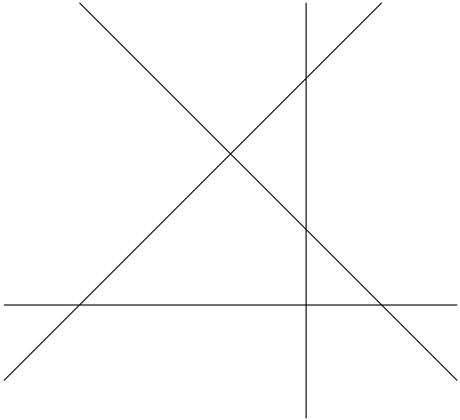

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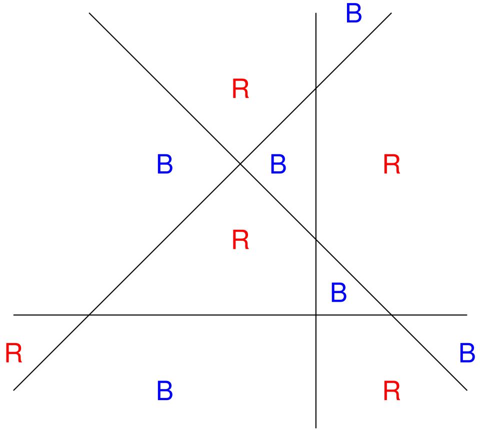

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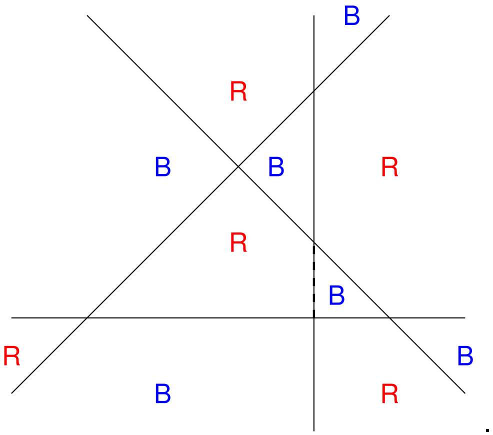

Fact: Swapping red and blue gives another valid colors.

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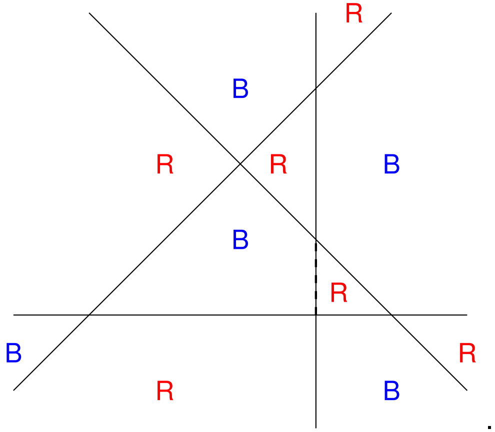

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Two color theorem: proof illustration.

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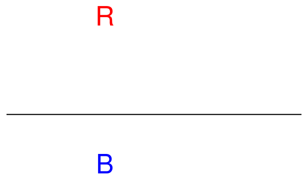

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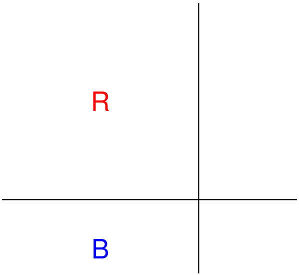
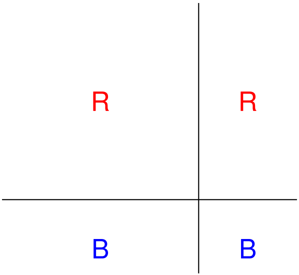


1. Add line.

Two color theorem: proof illustration.



1. Add line.
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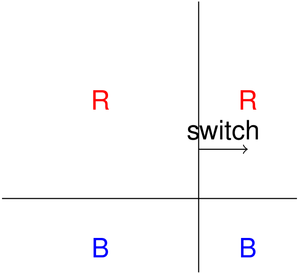

1. Add line.
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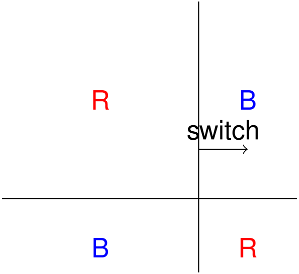
(Fixes conflicts along line, and makes no new ones.)

Two color theorem: proof illustration.



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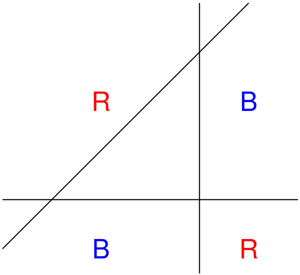

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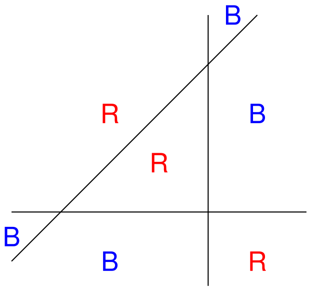

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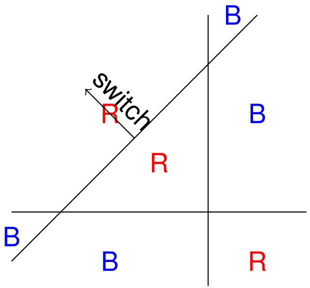

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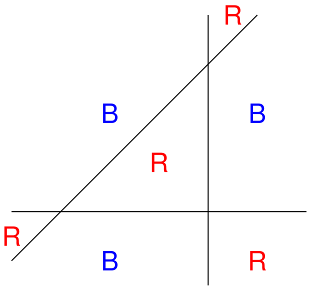

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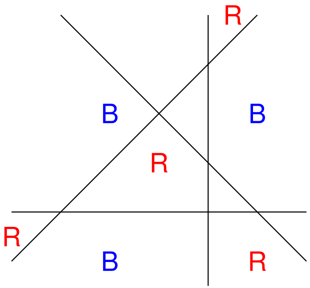

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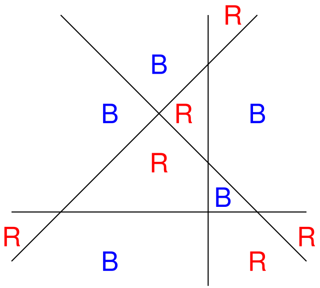

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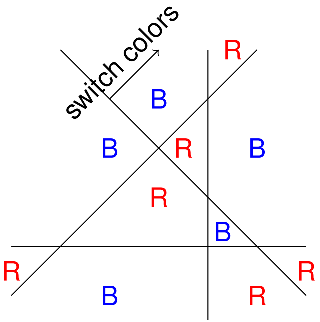

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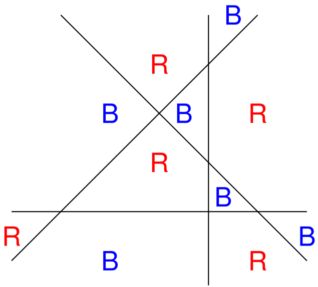

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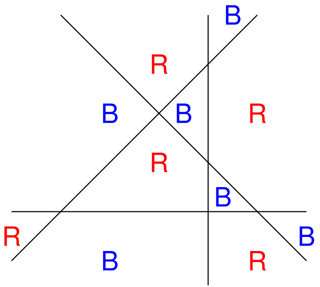

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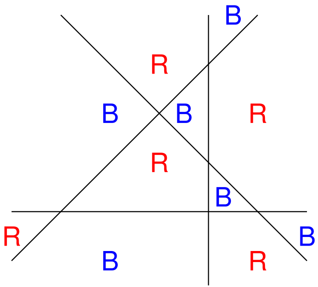
(Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$.

Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$. $\square$

Review Argument: $P(k) \implies P(k+1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.

Review Argument: $P(k) \implies P(k+1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.

For any “edge”.

Review Argument: $P(k) \implies P(k+1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.

For any “edge”.
Ok before switch.

Review Argument: $P(k) \implies P(k+1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.

For any "edge".
Ok before switch. Still ok, by "fact".

Review Argument: $P(k) \implies P(k+1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.

For any “edge”.
 Ok before switch. Still ok, by “fact”.
 Not ok before switch, must be on new line.

Review Argument: $P(k) \implies P(k+1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.

For any “edge”.
 Ok before switch. Still ok, by “fact”.
 Not ok before switch, must be on new line.
  Switch changes one side,

Review Argument: $P(k) \implies P(k+1)$.

Add line.
Inherit Colors.
Switch colors on one side of line.

For any “edge”.
Ok before switch. Still ok, by “fact”.
Not ok before switch, must be on new line.
Switch changes one side,
So now two sides have different colors.

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:
First Step

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:
  First Step (Base case).

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:
First Step (Base case).
Can step up the ladder of naturals.

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:
First Step (Base case).
Can step up the ladder of naturals. (Induction Step.)

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:
First Step (Base case).
Can step up the ladder of naturals. (Induction Step.)
Get to be on step k.

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:
  First Step (Base case).
  Can step up the ladder of naturals. (Induction Step.)
Get to be on step k. (Use induction hypothesis.)

Wrapup.

Proofs: Direct, By Contraposition, By Cases, By Contradiction.

Induction:
  First Step (Base case).
  Can step up the ladder of naturals. (Induction Step.)
Get to be on step k. (Use induction hypothesis.)

See you on Wednesday!