Continuous Probability (contd.)

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Multiple Continuous Random Variables

One defines a pair \((X, Y)\) of continuous RVs by specifying \(f_{X,Y}(x,y)\) for \(x, y \in \mathbb{R}\) where
\[
f_{X,Y}(x,y)\,dx\,dy = \Pr\{X \in (x, x + dx), Y \in (y, y + dy)\}.
\]
The function \(f_{X,Y}(x,y)\) is called the joint pdf of \(X\) and \(Y\).

Example: Choose a point \((X, Y)\) uniformly in the set \(A \subset \mathbb{R}^2\). Then
\[
f_{X,Y}(x,y) = \frac{1}{|A|} \mathbb{1}_{\{(x,y) \in A\}}
\]
where \(|A|\) is the area of \(A\).

Interpretation. Think of \((X, Y)\) as being discrete on a grid with mesh size \(\varepsilon\) and \(\Pr\{X = m\varepsilon, Y = n\varepsilon\} = f_{X,Y}(m\varepsilon,n\varepsilon)\varepsilon^2\).

Extension: \(X = (X_1, \ldots, X_n)\) with \(f_X(x)\).

Example of Continuous \((X, Y)\)

Pick a point \((X, Y)\) uniformly in the unit circle.

Expectation

Definitions: (a) The expectation of a random variable \(X\) with pdf \(f(x)\) is defined as
\[
E[X] = \int_{-\infty}^{\infty} x f(x)\,dx.
\]
(b) The expectation of a function of a random variable is defined as
\[
E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\,dx.
\]
(c) The expectation of a function of multiple random variables is defined as
\[
E[h(X_1, \ldots, X_n)] = \int \cdots \int h(x_1, x_2, \ldots, x_n) f_{X_1}(x_1) \cdots f_{X_n}(x_n)\,dx_1 \cdots dx_n.
\]

Justifications: Think of the discrete approximations of the continuous RVs.

Independent Continuous Random Variables

Definition: The continuous RVs \(X\) and \(Y\) are independent if
\[
\Pr\{X \in A, Y \in B\} = \Pr\{X \in A\} \Pr\{Y \in B\}, \quad \forall A, B.
\]

Theorem: The continuous RVs \(X\) and \(Y\) are independent if and only if
\[
f_{X,Y}(x,y) = f_{X}(x)f_{Y}(y).
\]

Proof: As in the discrete case.

Definition: The continuous RVs \(X_1, \ldots, X_n\) are mutually independent if
\[
\Pr\{X_1 \in A_1, \ldots, X_n \in A_n\} = \Pr\{X_1 \in A_1\} \cdots \Pr\{X_n \in A_n\}, \quad \forall A_1, \ldots, A_n.
\]

Theorem: The continuous RVs \(X_1, \ldots, X_n\) are mutually independent if and only if
\[
f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n).
\]

Proof: As in the discrete case.
**Examples of Independent Continuous RVs**

1. **Minimum of Independent Expo.** Let $X = \text{Expo}(\lambda)$ and $Y = \text{Expo}(\mu)$ be independent RVs.

   Recall that $Pr[X > u] = e^{-\lambda u}$. Then
   
   \[ Pr[\min\{X, Y\} > u] = Pr[X > u \land Y > u] = Pr[X > u]Pr[Y > u] = e^{-\lambda u}e^{-\mu u} = e^{-(\lambda + \mu)u}. \]

   This shows that $\min\{X, Y\} = \text{Expo}(\lambda + \mu)$.

   Thus, the minimum of two independent exponentially distributed RVs is exponentially distributed.

2. **Minimum of Independent $U[0, 1]$.** Let $X, Y \sim U[0, 1]$ be independent RVs. Let also $Z = \min\{X, Y\}$. What is $I_{Z}$?

   One has
   
   \[ Pr[Z > u] = Pr[X > u \land Y > u] = (1 - u)^2. \]

   Thus $F_Z(u) = Pr[Z \leq u] = 1 - (1 - u)^2$.

   Hence, $I_Z(u) = \frac{d}{du} F_Z(u) = 2(1 - u), u \in [0, 1]$. In particular,
   
   \[ E[Z] = \int_0^1 I_Z(u)du = \int_0^1 2(1 - u)du = 2 \frac{1}{2} - 2 \frac{1}{2} = \frac{1}{2}. \]

**Meeting at a Restaurant**

Two friends go to a restaurant independently uniformly at random between noon and 1pm. They agree they will wait for 10 minutes. When you put them together, they form a square with sides 5/6.

Thus, $Pr[\text{meet}] = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$.

**Expectation of Product of Independent RVs**

**Theorem** If $X, Y, Z$ are mutually independent, then


**Proof:** Same as discrete case.

**Example:** Let $X, Y, Z$ be mutually independent and $U[0, 1]$. Then

\[ E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 6XZ + 12YZ] \]

\[ = \frac{1}{3} + \frac{4}{9} + \frac{9}{4} + \frac{11}{2} + 6 \frac{11}{2} + 12 \frac{11}{2} \]

\[ = \frac{14}{3} \frac{22}{4} \approx 10.17. \]

**Variance**

**Definition:** The variance of a continuous random variable $X$ is defined as

\[ \text{var}[X] = E((X - E(X))^2) = E(X^2) - (E(X))^2. \]

**Example 1:** $X \sim U[0, 1]$. Then

\[ \text{var}[X] = E[X^2] - (E[X])^2 = \frac{1}{2} \frac{1}{4} = \frac{1}{12}. \]

**Example 2:** $X \sim \text{Expo}(\lambda)$. Then $E[X] = \lambda^{-1}$ and $E[X^2] = 2/(\lambda^2)$.

Hence, $\text{var}[X] = 1/(\lambda^2)$.

**Example 3:** Let $X, Y, Z$ be independent. Then

\[ \text{var}[X + Y + Z] = \text{var}[X] + \text{var}[Y] + \text{var}[Z], \]

as in the discrete case.

**Maximum of Two Exponentials**

Let $X = \text{Expo}(\lambda)$ and $Y = \text{Expo}(\mu)$ be independent. Define $Z = \max\{X, Y\}$.

Calculate $E[Z]$. We compute $I_Z$, then integrate.

One has

\[ Pr[Z < z] = Pr[X < z \land Y < z] = Pr[X < z]Pr[Y < z] = (1 - e^{-\lambda z})(1 - e^{-\mu z}) = 1 - e^{-\lambda z} - e^{-\mu z} + e^{-(\lambda + \mu)z}. \]

Thus,

\[ I_Z(z) = \lambda e^{-\lambda z} + \mu e^{-\mu z} - (\lambda + \mu) e^{-(\lambda + \mu)z}, \forall z > 0. \]

Hence,

\[ E[Z] = \int_0^\infty I_Z(z)dz = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda + \mu}. \]

**Meeting at a Restaurant**

Two friends go to a restaurant independently uniformly at random between noon and 1pm.

They agree they will wait for 10 minutes. What is the probability they meet?

Here, $(X, Y)$ are the times when the friends reach the restaurant.

The shaded area are the pairs where $|X - Y| < 1/6$, i.e., such that they meet.

The complement is the sum of two rectangles. When you put them together, they form a square with sides $5/6$.

Thus, $Pr[\text{meet}] = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$.

**Breaking a Stick**

You break a stick at two points chosen independently uniformly at random.

What is the probability you can make a triangle with the three pieces?

Let $X, Y$ be the two break points along the $[0, 1]$ stick.

You can make a triangle if $A < B < C < A + C$, and $C < B + A$.

If $X < Y$, this means $X < 0.5, Y < X < 0.5, Y > 0.5$. This is the blue triangle.

If $X > Y$, we get the red triangle, by symmetry.

Thus, $Pr[\text{make triangle}] = 1/4$. 
Maximum of \( n \) i.i.d. Exponentials

Let \( X_1, \ldots, X_n \) be i.i.d. Expo(1). Define \( Z = \max\{X_1, X_2, \ldots, X_n\} \).

Calculate \( E[Z] \).

We use a recursion. The key idea is as follows:

\[
Z = \min\{X_1, \ldots, X_n\} + V
\]

where \( V \) is the maximum of \( n - 1 \) i.i.d. Expo(1). This follows from the memoryless property of the exponential.

Let then \( A_0 = E[Z] \). We see that

\[
A_0 = E[\min\{X_1, \ldots, X_n\}] + A_{n-1}
\]

\[
= \frac{1}{n} + A_{n-1}
\]

because the minimum of Expo is Expo with the sum of the rates.

Hence,

\[
E[Z] = A_0 = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H(n).
\]

Geometric and Exponential

The geometric and exponential distributions are similar. They are both memoryless.

Consider flipping a coin every \( 1/N \) second with \( P[H] = p/N \), where \( N \gg 1 \).

Let \( X \) be the time until the first \( H \).

Fact: \( X \approx \text{Expo}(p) \).

Analysis: Note that

\[
P[X > t] = P[\text{first } Nt \text{ flips are tails}]
\]

\[
= (1 - \frac{p}{N})^{Nt} \approx \exp\{-pt\}.
\]

Indeed, \( (1 - \frac{a}{N})^N \approx \exp\{-a\} \).

Summary

Continuous Probability

- Continuous RVs are essentially the same as discrete RVs
- Think that \( X \approx x \) with probability \( f_X(x)\epsilon \)
- Sums become integrals, ...
- The exponential distribution is magical: memoryless.