CS70: Lecture 31.

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Gaussian RVs and CLT

1. Review: Continuous Probability: Geometric and Exponential
2. Normal Distribution
3. Central Limit Theorem
4. Examples

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7. Variance of Sum of Independent RVs: If X_n are pairwise independent, $var[X_1 + \cdots + X_n] = var[X_1] + \cdots + var[X_n]$

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Indeed, $(1 - \frac{a}{N})^N \approx \exp\{-a\}$.

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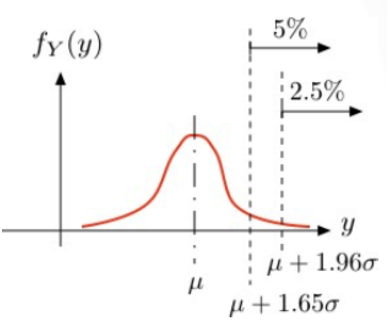

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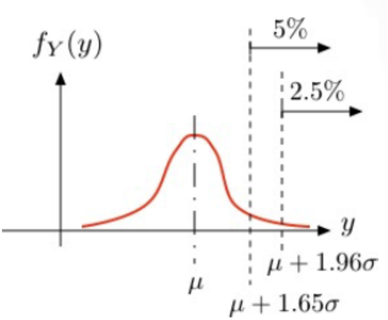

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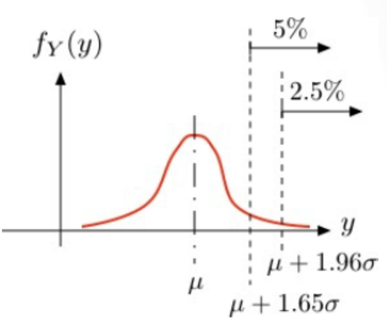


Note: $Pr[|Y-\mu| > 1.65\sigma] = 10\%$; $Pr[|Y-\mu| > 2\sigma] = 5\%$.

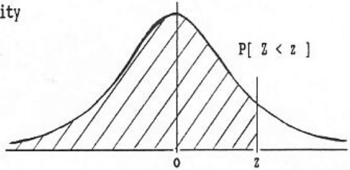
Table

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

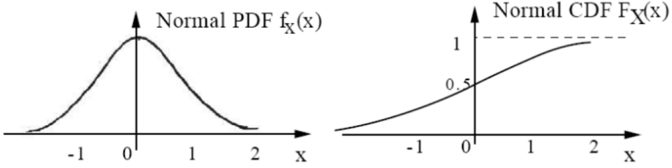
The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[\,Z < z\,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830

Standard Normal Variable



X is a standard normal variable if

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad x \in \Re$$

Since $f(x) = f(-x)$ and $\int_0^\infty x f_X(x) dx$ is finite, $E[X] = 0$.

Standard Normal Variable

$$var(X) = E[X^2] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{\frac{-x^2}{2}} dx$$

Integration by parts, $du = e^{\frac{-x^2}{2}}$, $v = x^2$

$$\begin{aligned} var(X) &= \frac{1}{\sqrt{2\pi}} (-xe^{\frac{-x^2}{2}})\Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx \\ &= 0 + 1 \end{aligned}$$

Bell Curve with zero mean and unit variance.

Standard Normal Variable (You're not responsible for this!)

We need to show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1$$

Use a trick: Let the value of the integral be A. Then

$$A^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$$

Now use polar co-ordinates.

$$A^2 = \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} e^{-\frac{r^2}{2}} rd\theta dr$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} 2\pi e^{-\frac{r^2}{2}} rdr$$

Substituting:

$$A^2 = -e^{\frac{r^2}{2}}]_0^{\infty} = 1$$

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Theorem Let $X = \mathcal{N}(0,1)$ and $Y = \mu + \sigma X$. Then

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$$\begin{aligned} f_Y(y) &= \frac{1}{\sigma} f_X(\frac{y-\mu}{\sigma}) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(y-\mu)^2}{2\sigma^2}\}. \quad \square \end{aligned}$$

Gaussian RV: Computing Probabilities

Example: Let the temperature of a city in the winter be X in Celsius. $X \sim N(2, 16)$. Find $P(-2 < X < 6)$.
Want to avoid computing the hairy integrals involved...Closed form of Normal CDF not available.
Procedure:

1. Convert to a standard normal: $\hat{X} = \frac{X-2}{4}$
2. Express in terms of probability problem of $\hat{X}$:

$$P(-2 < X < 6) = P(\frac{-2-2}{4} < \hat{X} < \frac{6-2}{4}) = P(-1 < \hat{X} < 1)$$

3. Look it up in a table of computed values of $F_{\hat{X}}(\hat{x})$
We want to find $P(\hat{X} < 1) - P(\hat{X} < -1) = \phi(1) - \phi(-1)$. Tables typically only list positive values, so $\phi(-1) = 1 - \phi(1)$ by symmetry.
4. Answer: $2\phi(1) - 1 =$

Note: $2\phi(k) - 1$ tells you how likely it is that the outcome is within k standard deviations. $k = 2 : 0.9544$;k=3: 0.9974

Crown Jewel of Normal Distribution

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Central Limit Theorem

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For any set of independent identically distributed (i.i.d.) random variables X_i, define $A_n = \frac{1}{n}\sum X_i$ to be the "running average" as a function of n.

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Note: We are asking this for **any arbitrary original distribution X_i!**

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Implications of CLT

1. The Distribution of S_n wipes out all the information in the original information except for μ and σ^2.
2. If there are large number of small and independent factors, the aggregate of these factors will be normally distributed. E.g. Noise.
3. The Gaussian Distribution is very important – many problems involve sums of iid random variables and the only thing one needs to know is the mean and variance.

Example: Guessing numbers

Alice picks 100 numbers uniformly from $[0,1]$. Bob must guess the sum within 2. What is the probability he is right if he guesses 55?
If X_i is Alice's i^{th} guess, it is difficult to calculate the CDF of $S_{100} = \sum_i X_i$.
$E[S_n] = 50, \mathrm{var}(S_n) = 100/12$.
So by the CLT, $\frac{S_{100}-50}{\sqrt{100/12}}$ is approx $N(0,1)$. Thus

$$P(53 \le S_{100} \le 57) \approx \phi(\frac{7}{2.887}) - \phi(\frac{3}{2.887}) = \phi(2.4247) - \phi(1.039)$$

$$= .9925 - .9251 = 0.0649$$

Example: Counting Customers

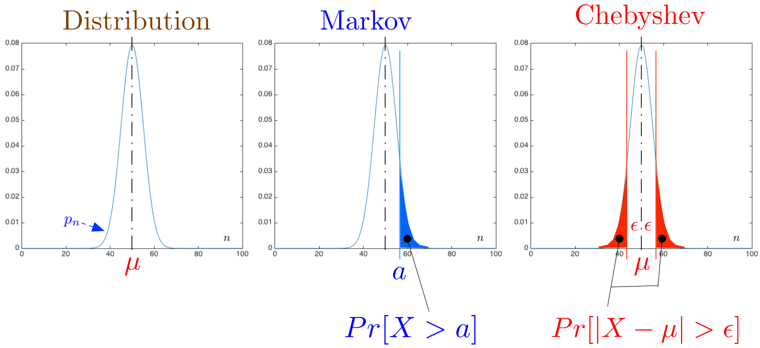
A store has one cashier who can serve one customer at a time.
The time taken to serve a customer $X \sim$ uniform on $[1, 5]$ minutes.
What is the probability that at least 200 customers can be served in 8 hours?
We want $P(\sum_{i=1}^{200} X_i \leq 480)$.
$E[S_{200}] = 600,\ var(S_{200}) = 200\ 16/12 = 800/3$.

$$P(200 \leq \sum_{i=1}^{200} X_i \leq 480) \approx \phi(\frac{480-600}{20\sqrt{2/3}}) - \phi(\frac{-400}{20\sqrt{2/3}})$$

$$= \phi(-6\sqrt{1.5}) - \phi(-20\sqrt{1.5}) = \phi(24.4949) - \phi(7.348).$$

$$P(\sum_{i=1}^{200} X_i \leq 480) \approx 0$$

Inequalities: A Preview



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