CS70: Lecture 32.

$\boxed{\text{Inequalities: Markov and Chebyshev}}$

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Inequalities: Markov and Chebyshev

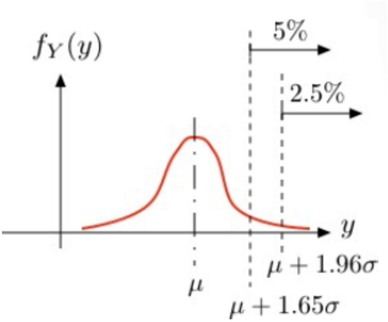
1. Review: Gaussian RV, CLT
2. Inequalities: Markov, Chebyshev
3. Examples
4. Confidence Intervals: Cheybshev Bound

Normal (Gaussian) Distribution.

For any μ and σ, a **normal** (aka **Gaussian**) random variable Y, which we write as $Y = \mathcal{N}(\mu, \sigma^2)$, has pdf

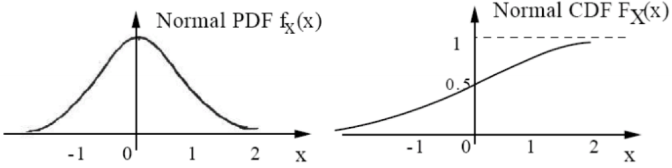
$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2}.$$

Standard normal has $\mu = 0$ and $\sigma = 1$.



Note: $Pr[|Y-\mu| > 1.65\sigma] = 10\%$; $Pr[|Y-\mu| > 2\sigma] = 5\%$.

Standard Normal Variable



X is a standard normal variable if

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad x \in \Re$$

Since $f(x) = f(-x)$ and $\int_0^{\infty} x f_X(x) dx$ is finite, $E[X] = 0$.

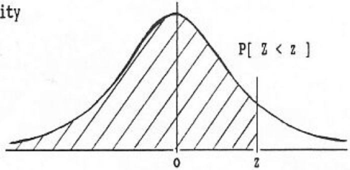
Table

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z
i.e.

$$P[\, Z < z \,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830

Recap: Crown Jewel of Normal Distribution

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Central Limit Theorem

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Note: We are asking this for **any arbitrary original distribution X_i!**

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That is,

$$Pr[S_n \le \alpha] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

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Implications of CLT

1. The Distributions of S_n and T_n wipe out all the information in the original information except for μ and σ^2.
2. If there are large number of small and independent factors, the aggregate of these factors will be normally distributed. E.g. Noise.
3. The Gaussian Distribution is very important – many problems involve sums of iid random variables and the only thing one needs to know is the mean and variance.

Example: Guessing numbers

Alice picks 100 numbers uniformly from $[0,1]$. Bob must guess the sum within 2. What is the probability he is right if he guesses 55?
If X_i is Alice's i^{th} guess, it is difficult to calculate the CDF of $S_{100} = \sum_i X_i$.
$E[S_n] = 50, \mathrm{var}(S_n) = 100/12$.
So by the CLT, $\frac{S_{100}-50}{\sqrt{100/12}}$ is approx $N(0,1)$. Thus

$$P(53 \leq S_{100} \leq 57) \approx \phi(\frac{7}{2.887}) - \phi(\frac{3}{2.887}) = \phi(2.4247) - \phi(1.039)$$

$$= .9925 - .9251 = 0.0649$$

Example: Counting Customers

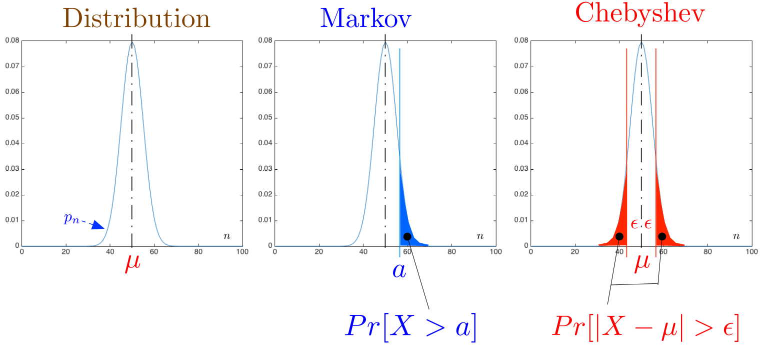
A store has one cashier who can serve one customer at a time.
The time taken to serve a customer $X \sim$ uniform on $[1, 5]$ minutes.
What is the probability that at least 200 customers can be served in 8 hours?
We want $P(\sum_{i=1}^{200} X_i \leq 480)$.
$E[S_{200}] = 600$, $var(S_{200}) = 200\ 16/12 = 800/3$.

$$P(200 \leq \sum_{i=1}^{200} X_i \leq 480) \approx \phi(\frac{480-600}{20\sqrt{2/3}}) - \phi(\frac{-400}{20\sqrt{2/3}})$$

$$= \phi(-6\sqrt{1.5}) - \phi(-20\sqrt{1.5}) = \phi(24.4949) - \phi(7.348).$$

$$P(\sum_{i=1}^{200} X_i \leq 480) \approx 0$$

Inequalities: An Overview



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Markov Inequality

If X can only take non-negative values then

$$P(X \geq a) \leq \frac{E[X]}{a}$$

for all $a > 0$.

This inequality makes no assumptions on the existence of variance and so it can't be very strong for typical distributions. In fact, it is quite weak.

Markov Inequality

Example: X is the height of a random adult in Berkeley. If $E[X] = 68$ inches, the Markov Inequality says that

$$P(X > 144) \leq \frac{68}{144} = 0.47$$

On the other hand since it is general, we can try to see what happens to it as we add more assumptions on the distribution. Think of this inequality as being the building block for others...

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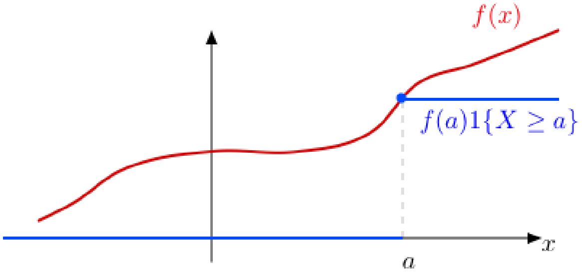
Taking the expectation yields the inequality, because expectation is monotone. $\square$

A picture



$$f(a)1\{X \geq a\} \leq f(x) \Rightarrow 1\{X \geq a\} \leq \frac{f(X)}{f(a)}$$

$$\Rightarrow Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$$

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$\square$

This result confirms that the variance measures the "deviations from the mean."

Chebyshev Inequality

If X is a random variable with finite mean and variance σ^2, then

$$P(|X - E[X]| \geq c) \leq \frac{\sigma^2}{c^2}$$

for all $c > 0$.
Also, letting $c = k\sigma$:

$$P(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

Example

Example: X is the height of a random adult in Berkeley. If $E[X] = 68$ inches, and $\sigma_X^2 = 49$, the Chebyshev Inequality says that

$$P(X > 144) \leq P(|X - 68| > 76) \leq \frac{49}{(76)^2} = 0.0084$$

Example: Random Walk

Starting at the origin, I flip a coin $10,000$ times: on each flip, if it is heads I step forward; if it is tails I step backward. Prob of heads is 0.5. Estimate the prob that I will be greater than 400 steps away from the origin.

For now let's say the coin is tossed n times. Let $X_i = 1$ if toss i is heads and -1 otherwise.

$E[X_i] = 0$ and $E[X_i^2] = 1$ so $var(X_i) = 1$

Let $X = \sum_i X_i$. $E[X] = 0$ and $var(X) = n$. We want $|X|$ since that's the magnitude of the distance.

Chebyshev says:

$$P(|X| \geq k\sqrt{n}) \leq \frac{1}{k^2}$$

$$P(|X| \geq 100k) \leq \frac{1}{k^2}$$

So, the prob I am more than 400 steps away is less than $\frac{1}{16}$.

Chebyshev Inequality

Example: We want to estimate $E[X]$ by measuring random adults and computing the average height (M_n). How many should we measure to ensure that the estimate is within 1 inch of $E[X]$ with probability 0.99? Assume that $\sigma_X^2 = 36$.
$var(M_n) = \frac{36}{n}$. We want

$$P(|M_n - E[M_n]| \leq 1) = 1 - P(|M_n - E[M_n]| > 1) \geq 0.99$$

$$P(|M_n - E[M_n]| > 1) \leq 0.01$$

$E[M_n] = E[X]$, so Chebyshev tells us that

$$P(|M_n - E[X]| \geq 1) \leq \frac{36}{n}$$

$$\frac{36}{n} \leq .01 \Rightarrow n \geq 3600.$$

In general to have 0.99 confidence that one is within c of the mean: $n \geq \frac{100\sigma^2}{c^2}$.

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$$Var(X_i) = p(1 - lp) \leq (.5)(.5) = \frac{1}{4}$$

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We will address WLLN next time.

Summary

$\boxed{\text{Inequalities: Markov and Chebyshev}}$

1. Inequalities: Markov and Chebyshev Tail Bounds
2. Confidence Intervals: Chebyshev Bounds