CS70: Lecture 33.

$\boxed{\text{WLLN, Confidence Intervals (CI): Chebyshev vs. CLT}}$

CS70: Lecture 33.

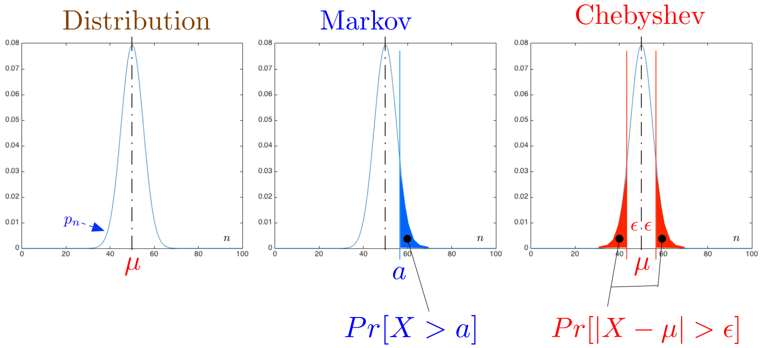
$\boxed{\text{WLLN, Confidence Intervals (CI): Chebyshev vs. CLT}}$

CS70: Lecture 33.

$\boxed{\text{WLLN, Confidence Intervals (CI): Chebyshev vs. CLT}}$

1. Review: Inequalities: Markov, Chebyshev
2. Law of Large Numbers
3. Review: CLT
4. Confidence Intervals: Chebyshev vs. CLT

Inequalities: An Overview



Markov Inequality

If X can only take non-negative values then

$$P(X \geq a) \leq \frac{E[X]}{a}$$

for all $a > 0$.

This inequality makes no assumptions on the existence of variance and so it can't be very strong for typical distributions. In fact, it is quite weak.

Chebyshev Inequality

If X is a random variable with finite mean and variance σ^2, then

$$P(|X - E[X]| \geq c) \leq \frac{\sigma^2}{c^2}$$

for all $c > 0$.
Also, letting $c = k\sigma$:

$$P(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

Fraction of *H*'s

Here is a classical application of Chebyshev's inequality.

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

By Chebyshev,

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

By Chebyshev,

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{var[M_n]}{(0.1)^2} = 100 var[M_n].$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

By Chebyshev,

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{var[M_n]}{(0.1)^2} = 100 var[M_n].$$

Now,

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

By Chebyshev,

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{var[M_n]}{(0.1)^2} = 100 var[M_n].$$

Now,

$$var[M_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n])$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

By Chebyshev,

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{var[M_n]}{(0.1)^2} = 100 var[M_n].$$

Now,

$$var[M_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n]) = \frac{1}{n} var[X_1]$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

By Chebyshev,

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{var[M_n]}{(0.1)^2} = 100 var[M_n].$$

Now,

$$var[M_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n]) = \frac{1}{n} var[X_1] \leq \frac{1}{4n}.$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|M_n - 0.5| \geq 0.1] = Pr[M_n \leq 0.4 \text{ or } M_n \geq 0.6].$$

By Chebyshev,

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{var[M_n]}{(0.1)^2} = 100 var[M_n].$$

Now,

$$var[M_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n]) = \frac{1}{n} var[X_1] \leq \frac{1}{4n}.$$

$$Var(X_i) = p(1-p) \leq (.5)(.5) = \frac{1}{4}$$

Fraction of H's

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

Fraction of H's

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

Fraction of *H*'s

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

Fraction of H's

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$,

Fraction of *H*'s

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$,

Fraction of H's

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of Hs is within $\varepsilon > 0$ of 50% approaches 1:

Fraction of H's

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of Hs is within $\varepsilon > 0$ of 50% approaches 1:

$$Pr[|M_n - 0.5| \leq \varepsilon] \to 1.$$

Fraction of *H*'s

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of *H*s is within $\varepsilon > 0$ of 50% approaches 1:

$$Pr[|M_n - 0.5| \leq \varepsilon] \to 1.$$

This is an example of the (Weak) Law of Large Numbers.

Fraction of *H*'s

$$M_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|M_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of Hs is within $\varepsilon > 0$ of 50% approaches 1:

$$Pr[|M_n - 0.5| \leq \varepsilon] \to 1.$$

This is an example of the (Weak) Law of Large Numbers.

We look at a general case next.

Weak Law of Large Numbers

We perform an experiment n times independently and

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The fact that $var(M_n) \to 0$ at rate $\frac{1}{n}$ is great but what does that tell us about $P(|M_n - E[X_i|)$? How quickly does it go to zero?
Just use Chebyshev: $P(|X - E[X]| \geq c) \leq \frac{\sigma^2}{c^2}$

$$P(|M_n - E[X_i]) \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

for any $\epsilon > 0$.
This is a form of the Weak Law of Large Numbers.

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ.

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $M_n = \frac{X_1 + \cdots + X_n}{n}$. Then

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proof:
Let $M_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$Pr[|M_n - \mu| \geq \varepsilon] \quad \leq \quad \frac{var[M_n]}{\varepsilon^2}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $M_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$Pr[|M_n - \mu| \geq \varepsilon] \quad \leq \quad \frac{var[M_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2 \varepsilon^2}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $M_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$\begin{aligned} Pr[|M_n - \mu| \geq \varepsilon] &\leq \frac{var[M_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2 \varepsilon^2} \\ &= \frac{n var[X_1]}{n^2 \varepsilon^2} \end{aligned}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $M_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$\begin{aligned} Pr[|M_n - \mu| \geq \varepsilon] &\leq \frac{var[M_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2 \varepsilon^2} \\ &= \frac{n var[X_1]}{n^2 \varepsilon^2} = \frac{var[X_1]}{n \varepsilon^2} \end{aligned}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $M_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$\begin{aligned} Pr[|M_n - \mu| \geq \varepsilon] &\leq \frac{var[M_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2 \varepsilon^2} \\ &= \frac{n var[X_1]}{n^2 \varepsilon^2} = \frac{var[X_1]}{n \varepsilon^2} \to 0, \text{ as } n \to \infty. \end{aligned}$$

$\square$

What does the Weak Law Really Mean?

WLLN: $\lim_{n\to\infty} P(|M_n - \mu| \geq \epsilon) = 0$.
Just using the defn of limit: For any $\epsilon, \delta > 0$, there exists a number $n(\epsilon, \delta)$ such that

$$P(|M_n - \mu| \geq \epsilon) \leq \delta \quad \text{for all } n \geq n(\epsilon, \delta)$$

- δ:Confidence level
- ϵ: "Error"
- $n(\epsilon, \delta)$: threshold function for a given level of confidence and accuracy

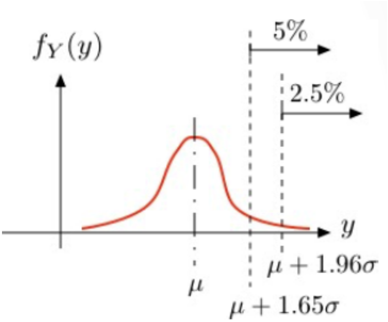
What this is saying is that if we compute M_n for large n then:
Almost Always, $|M_n - \mu| < \epsilon$.
We say that M_n **converges to** μ **in probability**.

Recap: Normal (Gaussian) Distribution.

For any μ and σ, a **normal** (aka **Gaussian**) random variable Y, which we write as $Y = \mathcal{N}(\mu, \sigma^2)$, has pdf

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2}.$$

Standard normal has $\mu = 0$ and $\sigma = 1$.



Note: $Pr[|Y - \mu| > 1.65\sigma] = 10\%$; $Pr[|Y - \mu| > 2\sigma] = 5\%$.

Recap: Central Limit Theorem

Recap: Central Limit Theorem

Central Limit Theorem
Let $X_1, X_2, \dots$ be i.i.d. with $E[X_1] = \mu$ and $var(X_1) = \sigma^2$. Define

$$S_n := \frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}.$$

Then,

$$S_n \to \mathcal{N}(0,1), \text{as } n \to \infty.$$

That is,

$$Pr[S_n \leq \alpha] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2} dx.$$

$$E(S_n) = \frac{1}{\sigma/\sqrt{n}} (E(A_n) - \mu) = 0$$

$$Var(S_n) = \frac{1}{\sigma^2/n} Var(A_n) = 1.$$

Confidence Interval (CI) for Mean: CLT

Confidence Interval (CI) for Mean: CLT

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2.

Confidence Interval (CI) for Mean: CLT

Let $X_1, X_2, \dots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

Confidence Interval (CI) for Mean: CLT

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Confidence Interval (CI) for Mean: CLT

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Thus, for $n \gg 1$, one has

$$Pr[-2 \leq (\frac{A_n - \mu}{\sigma/\sqrt{n}}) \leq 2] \approx 95\%.$$

Confidence Interval (CI) for Mean: CLT

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Thus, for $n \gg 1$, one has

$$Pr[-2 \leq (\frac{A_n - \mu}{\sigma/\sqrt{n}}) \leq 2] \approx 95\%.$$

Equivalently,

$$Pr[\mu \in [A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}]] \approx 95\%.$$

Confidence Interval (CI) for Mean: CLT

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{A_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Thus, for $n \gg 1$, one has

$$Pr[-2 \le (\frac{A_n - \mu}{\sigma/\sqrt{n}}) \le 2] \approx 95\%.$$

Equivalently,

$$Pr[\mu \in [A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}]] \approx 95\%.$$

That is,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.$$

CI for Mean: CLT vs. Chebyshev

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Also,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.$$

CI for Mean: CLT vs. Chebyshev

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Also,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.$$

What would Chebyshev's bound give us?

CI for Mean: CLT vs. Chebyshev

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Also,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.$$

What would Chebyshev's bound give us?

$$[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.$$

CI for Mean: CLT vs. Chebyshev

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Also,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.$$

What would Chebyshev's bound give us?

$$[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.(Why?)$$

CI for Mean: CLT vs. Chebyshev

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance σ^2. Let

$$A_n = \frac{X_1 + \cdots + X_n}{n}.$$

The CLT states that

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

Also,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.$$

What would Chebyshev's bound give us?

$$[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu.(Why?)$$

Thus, the CLT provides a smaller confidence interval.

Coins and CLT.

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$.

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n, p)$.

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.

Here, $\mu = p$

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.

Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$.

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.

Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

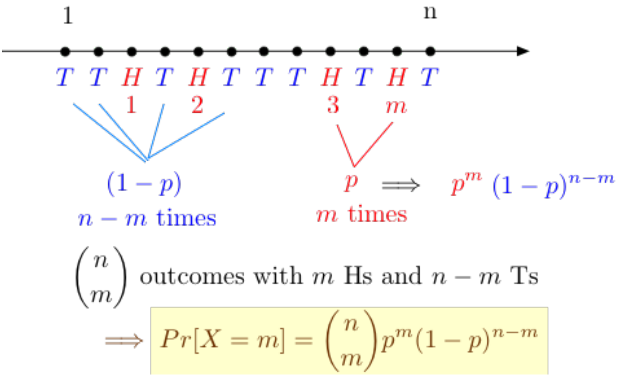
$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1).$$

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.

Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1).$$



Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$. Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1)$$

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.
Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1)$$

and

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu$$

with $A_n = (X_1 + \cdots + X_n)/n$.

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.
Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1)$$

and

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu$$

with $A_n = (X_1 + \cdots + X_n)/n$.
Hence,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } p.$$

Coins and CLT.

Let $X_1, X_2, \dots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.
Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1)$$

and

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu$$

with $A_n = (X_1 + \cdots + X_n)/n$.
Hence,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } p.$$

Since $\sigma \leq 0.5$,

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.
Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1)$$

and

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu$$

with $A_n = (X_1 + \cdots + X_n)/n$.
Hence,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } p.$$

Since $\sigma \leq 0.5$,

$$[A_n - 2\frac{0.5}{\sqrt{n}}, A_n + 2\frac{0.5}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } p.$$

Coins and CLT.

Let $X_1, X_2, \ldots$ be i.i.d. $B(p)$. Thus, $X_1 + \cdots + X_n = B(n,p)$.
Here, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. CLT states that

$$\frac{X_1 + \cdots + X_n - np}{\sqrt{p(1-p)n}} \to \mathcal{N}(0,1)$$

and

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } \mu$$

with $A_n = (X_1 + \cdots + X_n)/n$.
Hence,

$$[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } p.$$

Since $\sigma \leq 0.5$,

$$[A_n - 2\frac{0.5}{\sqrt{n}}, A_n + 2\frac{0.5}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } p.$$

Thus,

$$[A_n - \frac{1}{\sqrt{n}}, A_n + \frac{1}{\sqrt{n}}] \text{ is a } 95\% - \text{CI for } p.$$

Comparing Chebyshev and CLT: Polling

We ask n randomly sampled voters whether they support Bob.
$X_i = 1$ if the i^{th} voter says "yes" and $X_i = 0$ otherwise. The X_i are iid.
We want to be sure with prob ≥ 0.95 that $|M_{100} - p| \leq 0.1$. How many people should we ask?
Again, use the bound that $\mathrm{var}(X_i) \leq \frac{1}{4}$
By Chebyshev:

$$\frac{25}{n} \leq 0.05 \Rightarrow \boxed{n \geq 500}$$

By CLT:

$$2(1 - \phi(2 * 0.1 * \sqrt{n})) \leq 0.05$$

$$\phi(2 * 0.1 * \sqrt{n}) \geq 0.975$$

Since $\phi(1.96) = 0.975$:

$$\boxed{n \geq 96.04}$$

CLT much better than Chebyshev.

Summary

$\boxed{\text{Inequalities and Confidence Interals}}$

1. Inequalities: Markov and Chebyshev Tail Bounds
2. Weak Law of Large Numbers
3. Confidence Intervals: Chebyshev Bounds vs. CLT Approx.
4. CLT: X_n i.i.d. $\implies \frac{A_n - \mu}{\sigma/\sqrt{n}} \rightarrow \mathcal{N}(0,1)$

Summary

Inequalities and Confidence Interals

1. Inequalities: Markov and Chebyshev Tail Bounds
2. Weak Law of Large Numbers
3. Confidence Intervals: Chebyshev Bounds vs. CLT Approx.
4. CLT: X_n i.i.d. $\Longrightarrow \frac{A_n - \mu}{\sigma/\sqrt{n}} \to \mathcal{N}(0,1)$
5. CI: $[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] =$ 95%-CI for μ.

Summary

$\boxed{\text{Inequalities and Confidence Interals}}$

1. Inequalities: Markov and Chebyshev Tail Bounds
2. Weak Law of Large Numbers
3. Confidence Intervals: Chebyshev Bounds vs. CLT Approx.
4. CLT: X_n i.i.d. $\Longrightarrow \frac{A_n - \mu}{\sigma/\sqrt{n}} \to \mathcal{N}(0,1)$
5. CI: $[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] = 95\%$-CI for μ.