CS70: Lecture 34.

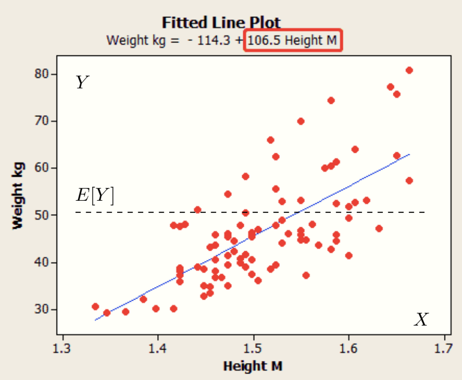
Linear Regression (LR)

1. Motivation for Linear Regression (LR)
2. Minimum Mean Squared Error: Discussion
3. Covariance: Definition and Properties
4. Linear Regression (LR): Non-Bayesian vs. Bayesian (LLSE)
5. Derivation and Illustration

Linear Regression: Motivation

Example 1: 100 people.

Let (X_n, Y_n) = (height, weight) of person n, for $n = 1, \ldots, 100$:



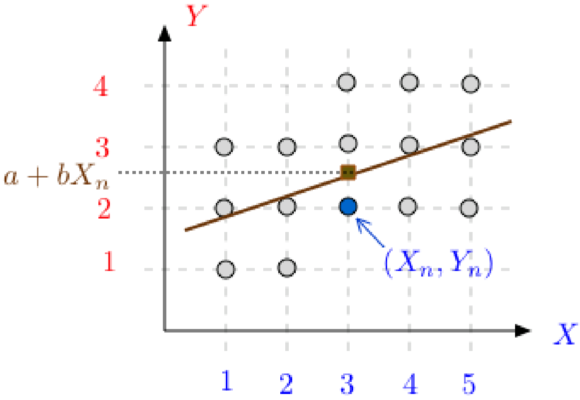
The blue line is $Y = -114.3 + 106.5X$. (X in meters, Y in kg.)

Best linear fit: Linear Regression.

Motivation

Example 2: 15 people.

We look at two attributes: (X_n, Y_n) of person n, for $n = 1, \ldots, 15$:



The line $Y = a + bX$ is the linear regression.

Linear Regression: Discussion

If we want to guess the value of a random variable Y, and know nothing more than its distributon, what's our best guess?

Depends on how we measure the 'goodness' of our guess.

Say we use the **expected squared error between Y and our guess** as the "error" measure. Then? Answer is: $E[Y]$.

More precisely, the value of a that minimizes $E[(Y-a)^2]$ is $a = E[Y]$.

Proof:

Let $\hat{Y} := Y - E[Y]$. Then, $E[\hat{Y}] = 0$. So, $E[\hat{Y}c] = 0, \forall c$. Now,

$$\begin{aligned}
E[(Y-a)^2] &= E[(Y - E[Y] + E[Y] - a)^2] \\
&= E[(\hat{Y} + c)^2] \text{ with } c = E[Y] - a \\
&= E[\hat{Y}^2 + 2\hat{Y}c + c^2] = E[\hat{Y}^2] + 2E[\hat{Y}c] + c^2 \\
&= E[\hat{Y}^2] + 0 + c^2 \geq E[\hat{Y}^2].
\end{aligned}$$

Hence, $E[(Y-a)^2] \geq E[(Y - E[Y])^2], \forall a$. □

Linear Regression: Discussion

Thus, if we want to guess the value of Y, we choose $E[Y]$.

Now assume we make some observation X related to Y.

How do we use that observation to improve our guess about Y?

Idea: use a function $g(X)$ of the observation to estimate Y.

The simplest $g(X)$ is a constant that does not depend on X.

The next simplest function is linear: $g(X) = a + bX$.

What is the best linear function? That is our next topic.

(We can also consider a general function $g(X)$. Any guess on what is the best function to use? Answer: $E[Y|X]$.)

Covariance

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Fact

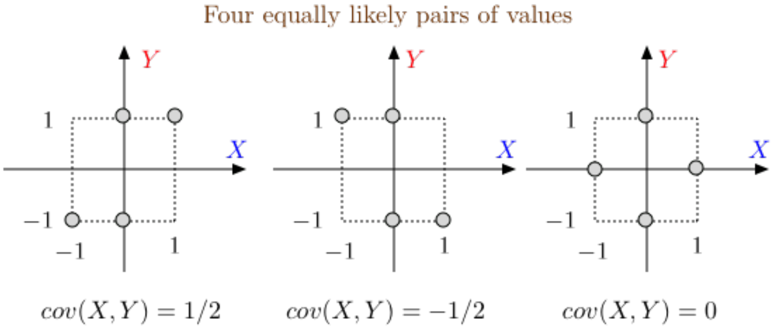
$$cov(X, Y) = E[XY] - E[X]E[Y].$$

Proof:

$$\begin{aligned} E[(X - E[X])(Y - E[Y])] &= E[XY - E[X]Y - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]. \end{aligned}$$

□

Examples of Covariance



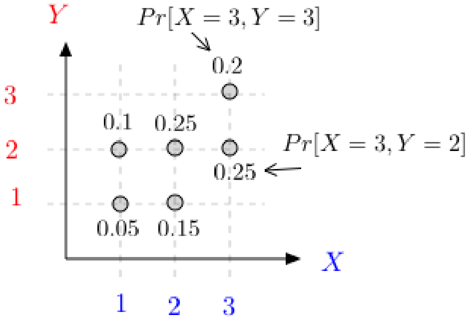
Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $cov(X, Y) = E[XY]$.

When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together. X and Y are said to be positively correlated.

When $cov(X, Y) < 0$, when X is larger, Y tends to be smaller. X and Y are said to be negatively correlated.

When $cov(X, Y) = 0$, we say that X and Y are uncorrelated.

Examples of Covariance



$$E[X] = 1 \times 0.15 + 2 \times 0.4 + 3 \times 0.45 = 1.9$$
$$E[X^2] = 1^2 \times 0.15 + 2^2 \times 0.4 + 3^2 \times 0.45 = 5.8$$
$$E[Y] = 1 \times 0.2 + 2 \times 0.6 + 3 \times 0.2 = 2$$
$$E[XY] = 1 \times 0.05 + 1 \times 2 \times 0.1 + \cdots + 3 \times 3 \times 0.2 = 4.85$$
$$cov(X, Y) = E[XY] - E[X]E[Y] = 1.05$$
$$var[X] = E[X^2] - E[X]^2 = 2.19.$$

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X,X)$

(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a+X, b+Y) = cov(X,Y)$

(d) $cov(aX+bY, cU+dV) = ac.cov(X,U) + ad.cov(X,V) + bc.cov(Y,U) + bd.cov(Y,V).$

Proof:

Prove (a),(b),(c) yourself to check your understanding.

(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean. Then,

$$\begin{aligned} cov(aX+bY, cU+dV) &= E[(aX+bY)(cU+dV)] \\ &= ac.E[XU] + ad.E[XV] + bc.E[YU] + bd.E[YV] \\ &= ac.cov(X,U) + ad.cov(X,V) + bc.cov(Y,U) + bd.cov(Y,V). \end{aligned}$$

□

Linear Regression: Non-Bayesian

Definition
Given the samples $\{(X_n, Y_n), n = 1, \ldots, N\}$, the Linear Regression of Y over X is

$$\hat{Y} = a + bX$$

where (a, b) minimize

$$\sum_{n=1}^{N} (Y_n - a - bX_n)^2.$$

Thus, $\hat{Y}_n = a + bX_n$ is our guess about Y_n given X_n. The squared error is $(Y_n - \hat{Y}_n)^2$. The LR minimizes the sum of the squared errors.

Note: This is a non-Bayesian formulation: there is no prior.

Linear Least Squares Estimate

Definition
Given two RVs X and Y with known distribution $Pr[X = x, Y = y]$, the Linear Least Squares Estimate of Y given X is

$$\hat{Y} = a + bX =: L[Y|X]$$

where (a,b) minimize

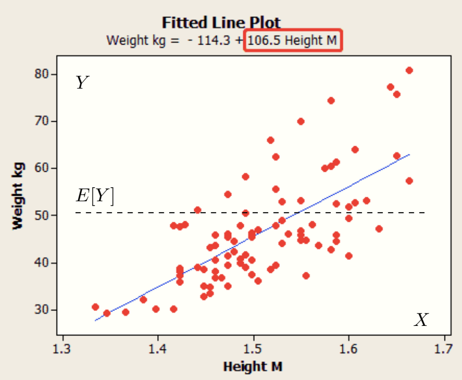
$$g(a,b) := E[(Y - a - bX)^2].$$

Thus, $\hat{Y} = a + bX$ is our guess about Y given X. The squared error is $(Y - \hat{Y})^2$. The LLSE minimizes the expected value of the squared error.

Note: This is a Bayesian formulation: there is a prior.

Linear Regression: Example

Example 1: 100 people.
Let (X_n, Y_n) = (height, weight) of person n, for $n = 1, \ldots, 100$:



The blue line is $Y = -114.3 + 106.5X$. (X in meters, Y in kg.) Best linear fit: Linear Regression.

LR: Non-Bayesian or Uniform?

Observe that

$$\frac{1}{N}\sum_{n=1}^{N}(Y_n - a - bX_n)^2 = E[(Y - a - bX)^2]$$

where one assumes that

$$(X, Y) = (X_n, Y_n), \text{ w.p. } \frac{1}{N} \text{ for } n = 1, \ldots, N.$$

That is, the non-Bayesian LR is equivalent to the Bayesian LLSE that assumes that (X, Y) is uniform on the set of observed samples.

Thus, we can study the two cases LR and LLSE in one shot.

However, the interpretations are different!

LLSE

Theorem
Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$. Then,

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

Proof 1:
$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X])$. Hence, $E[Y - \hat{Y}] = 0$.

Also, $E[(Y - \hat{Y})X] = 0$, after a bit of algebra. (See next slide.)

Hence, by combining the two brown equalities, $E[(Y - \hat{Y})(c + dX)] = 0$. Then, $E[(Y - \hat{Y})(\hat{Y} - a - bX)] = 0, \forall a, b$.
Indeed: $\hat{Y} = \alpha + \beta X$ for some α, β, so that $\hat{Y} - a - bX = c + dX$ for some c, d. Now,

$$E[(Y - a - bX)^2] = E[(Y - \hat{Y} + \hat{Y} - a - bX)^2]$$
$$= E[(Y - \hat{Y})^2] + E[(\hat{Y} - a - bX)^2] + 0 \geq E[(Y - \hat{Y})^2].$$

This shows that $E[(Y - \hat{Y})^2] \leq E[(Y - a - bX)^2]$, for all (a, b).
Thus $\hat{Y}$ is the LLSE. $\square$

A Bit of Algebra

$$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X]).$$

Hence, $E[Y - \hat{Y}] = 0$. We want to show that $E[(Y - \hat{Y})X] = 0$.

Note that

$$E[(Y - \hat{Y})X] = E[(Y - \hat{Y})(X - E[X])],$$

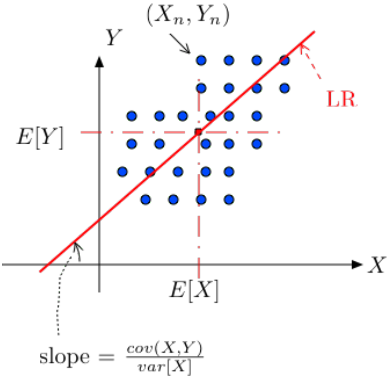
because $E[(Y - \hat{Y})E[X]] = 0$.

Now,

$$\begin{aligned} & E[(Y - \hat{Y})(X - E[X])] \\ & \quad = E[(Y - E[Y])(X - E[X])] - \frac{cov(X,Y)}{var[X]} E[(X - E[X])(X - E[X])] \\ & \quad =^{(*)} cov(X,Y) - \frac{cov(X,Y)}{var[X]} var[X] = 0. \quad \square \end{aligned}$$

$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

LR: Illustration



Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

Summary

Linear Regression

1. Covariance: $cov(X,Y) := E[(X-E[X])(Y-E[Y])]$.
2. Linear Regression: $L[Y|X] = E[Y] + \frac{cov(X,Y)}{var(X)}(X-E[X])$
3. Non-Bayesian: minimize $\sum_n (Y_n - a - bX_n)^2$
4. Bayesian: minimize $E[(Y-a-bX)^2]$