CS70: Lecture 34.

Linear Regression (LR)

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1. Motivation for Linear Regression (LR)
2. Minimum Mean Squared Error: Discussion
3. Covariance: Definition and Properties
4. Linear Regression (LR): Non-Bayesian vs. Bayesian (LLSE)
5. Derivation and Illustration

Linear Regression: Motivation

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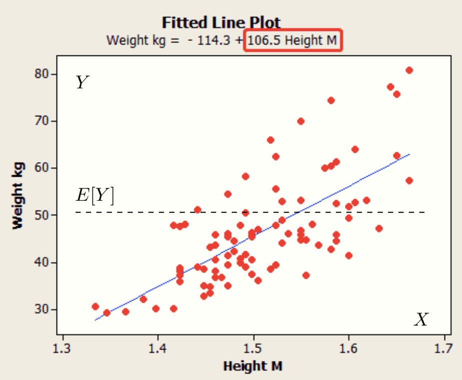

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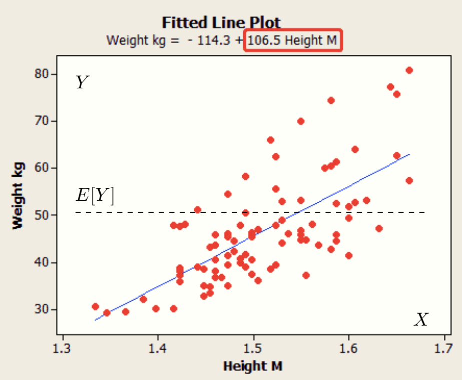

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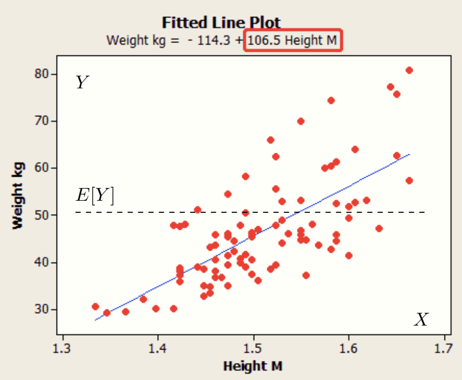

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Best linear fit: Linear Regression.

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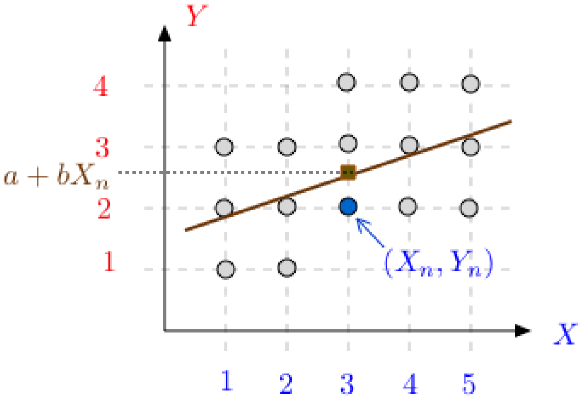

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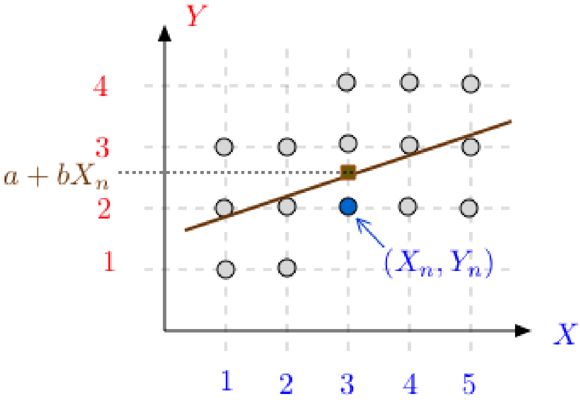

The line $Y = a + bX$ is the linear regression.

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Hence, $E[(Y-a)^2] \geq E[(Y - E[Y])^2], \forall a$. □

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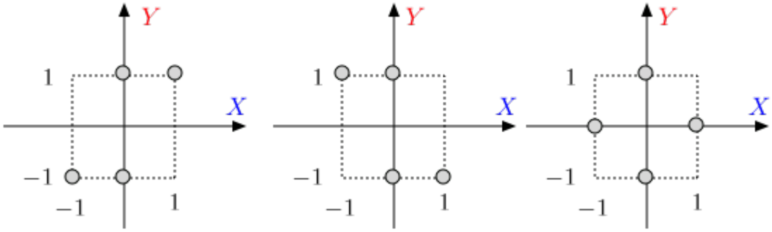
□

Examples of Covariance

Four equally likely pairs of values



$cov(X,Y) = 1/2$ $\qquad$ $cov(X,Y) = -1/2$ $\qquad$ $cov(X,Y) = 0$

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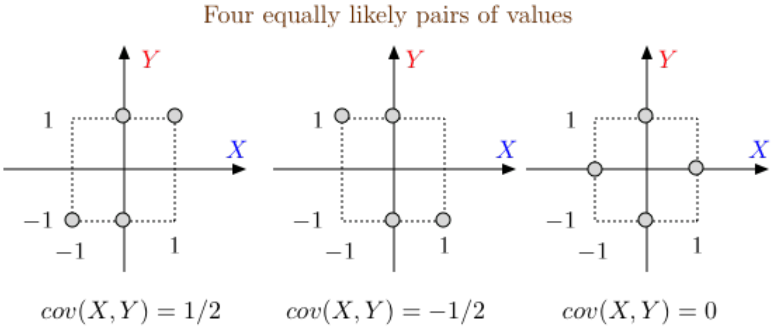

Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $cov(X, Y) = E[XY]$.

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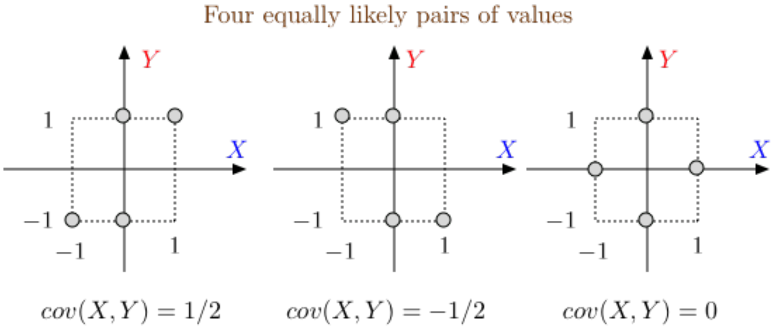

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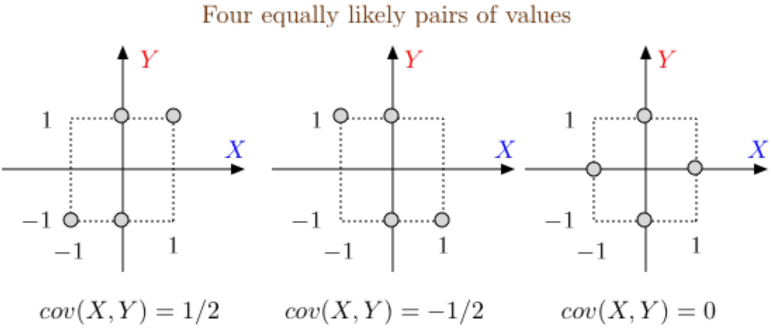

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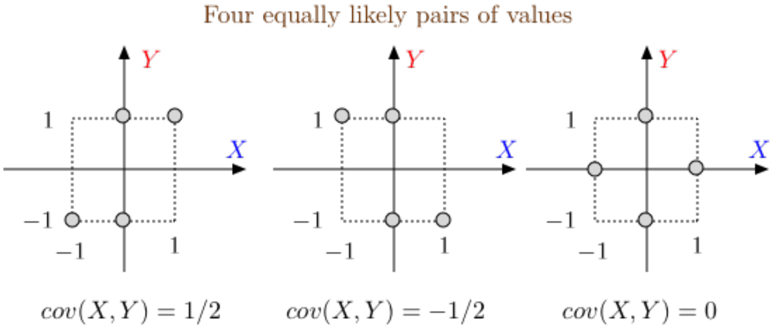

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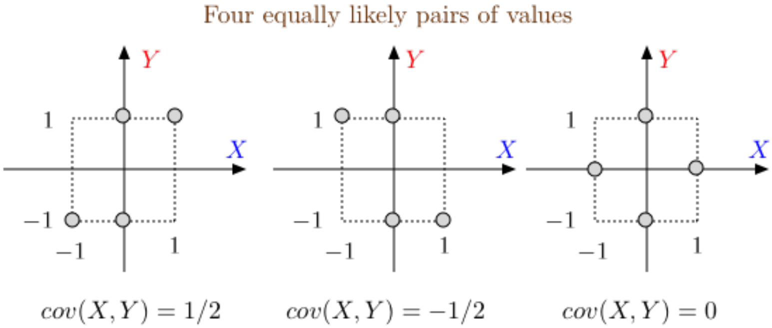

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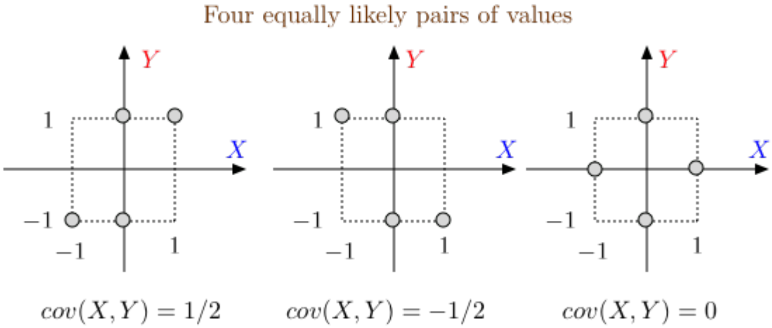

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When $cov(X, Y) = 0$, we say that X and Y are uncorrelated.

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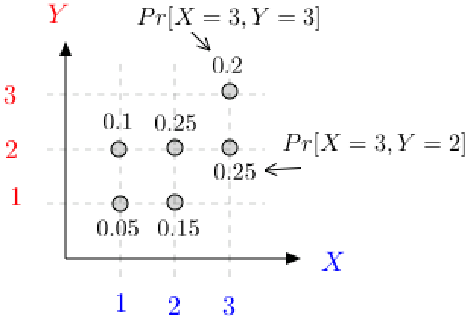

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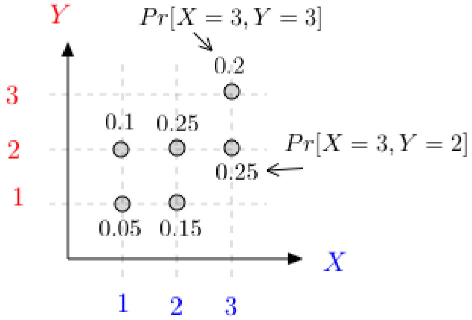

$$E[X] = 1 \times 0.15 + 2 \times 0.4 + 3 \times 0.45 = 1.9$$

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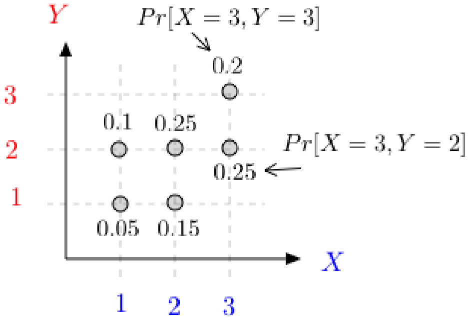

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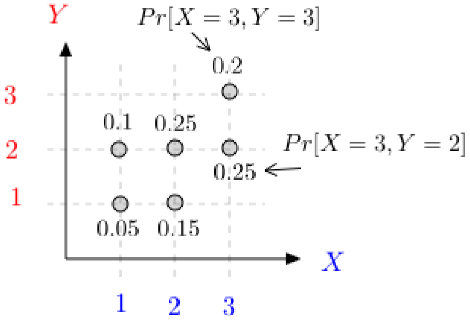

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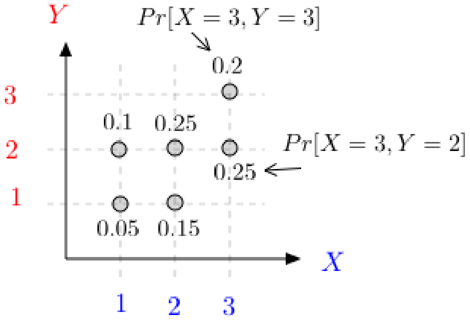
Examples of Covariance



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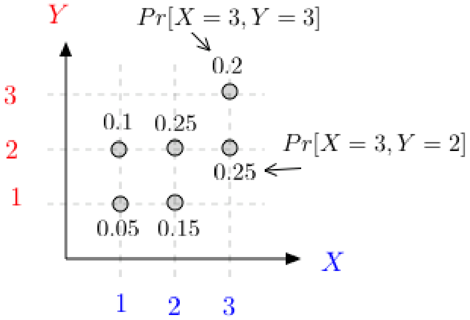

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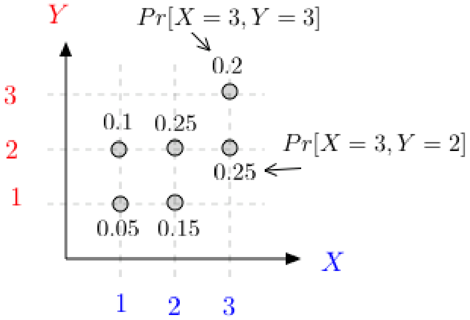

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Linear Regression: Example

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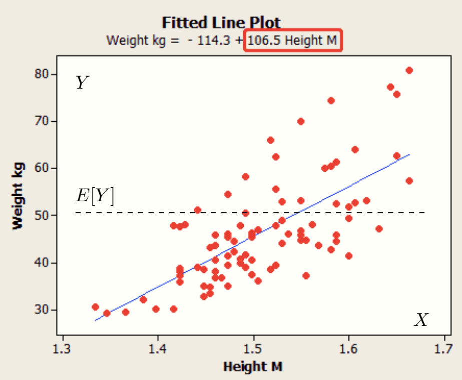
Example 1: 100 people.
Let (X_n, Y_n) = (height, weight) of person n, for $n = 1, \ldots, 100$:



The blue line is $Y = -114.3 + 106.5X$. (X in meters, Y in kg.) Best linear fit: Linear Regression.

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However, the interpretations are different!

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Hence, by combining the two brown equalities, $E[(Y - \hat{Y})(c + dX)] = 0$. Then, $E[(Y - \hat{Y})(\hat{Y} - a - bX)] = 0, \forall a, b$. Indeed: $\hat{Y} = \alpha + \beta X$ for some α, β, so that $\hat{Y} - a - bX = c + dX$ for some c, d. Now,

$$\begin{aligned} E[(Y - a - bX)^2] &= E[(Y - \hat{Y} + \hat{Y} - a - bX)^2] \\ &= E[(Y - \hat{Y})^2] + E[(\hat{Y} - a - bX)^2] + 0 \geq E[(Y - \hat{Y})^2]. \end{aligned}$$

This shows that $E[(Y - \hat{Y})^2] \leq E[(Y - a - bX)^2]$, for all (a, b).

LLSE

Theorem
Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$. Then,

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

Proof 1:
$Y - \hat{Y} = (Y - E[Y]) - \frac{cov(X,Y)}{var[X]}(X - E[X])$. Hence, $E[Y - \hat{Y}] = 0$.

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Thus $\hat{Y}$ is the LLSE. $\square$

A Bit of Algebra

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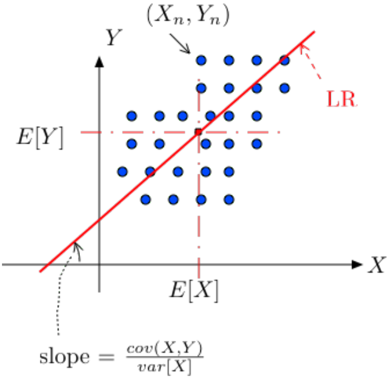
$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

LR: Illustration



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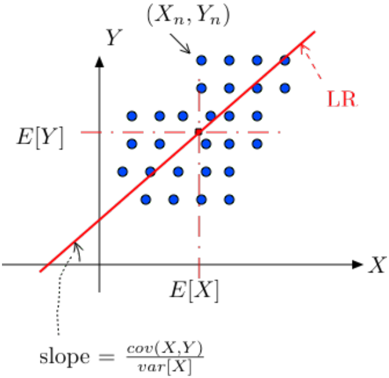

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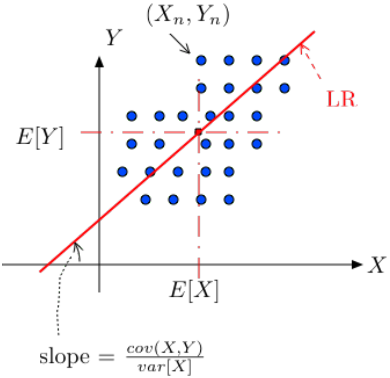

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- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

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4. Bayesian: minimize $E[(Y-a-bX)^2]$