CS70: Lecture 35.

Regression (contd.): Linear and Beyond

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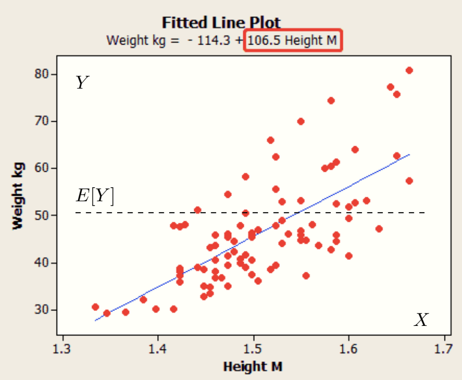
Regression (contd.): Linear and Beyond

1. Review: Linear Regression (LR), LLSE
2. LR: Examples
3. Beyond LR: Quadratic Regression
4. Conditional Expectation (CE) and properties
5. Non-linear Regression: CE = Minimum Mean-Squared Error (MMSE)

Review: Linear Regression – Motivation

Example: 100 people.
Let (X_n, Y_n) = (height, weight) of person n, for $n = 1, \ldots, 100$:



The blue line is $Y = -114.3 + 106.5X$. (X in meters, Y in kg.) Best linear fit: Linear Regression.

Review: Covariance

Definition

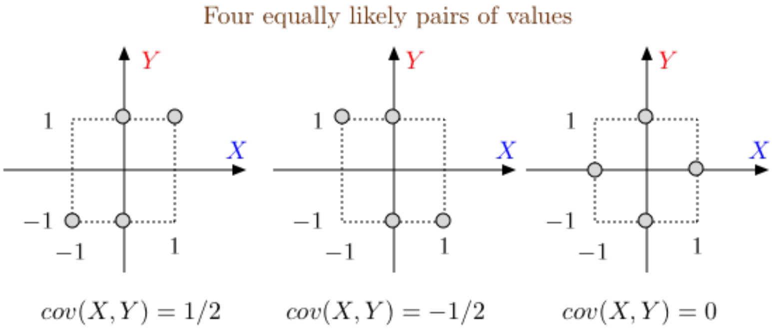
The covariance of X and Y is

$$cov(X,Y) := E[(X - E[X])(Y - E[Y])].$$

Fact

$$cov(X,Y) = E[XY] - E[X]E[Y].$$

Review: Examples of Covariance



Note that $E[X] = 0$ and $E[Y] = 0$ in these examples. Then $cov(X, Y) = E[XY]$.

When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together. X and Y are said to be positively correlated.

When $cov(X, Y) < 0$, when X is larger, Y tends to be smaller. X and Y are said to be negatively correlated.

When $cov(X, Y) = 0$, we say that X and Y are uncorrelated.

Review: Linear Regression – Non-Bayesian

Definition
Given the samples $\{(X_n, Y_n), n = 1, \ldots, N\}$, the Linear Regression of Y over X is

$$\hat{Y} = a + bX$$

where (a, b) minimize

$$\sum_{n=1}^{N} (Y_n - a - bX_n)^2.$$

Thus, $\hat{Y}_n = a + bX_n$ is our guess about Y_n given X_n. The squared error is $(Y_n - \hat{Y}_n)^2$. The LR minimizes the sum of the squared errors. Note: This is a non-Bayesian formulation: there is no prior.

Review: Linear Least Squares Estimate (LLSE)

Definition
Given two RVs X and Y with known distribution $Pr[X = x, Y = y]$, the Linear Least Squares Estimate of Y given X is

$$\hat{Y} = a + bX =: L[Y|X]$$

where (a, b) minimize

$$g(a,b) := E[(Y - a - bX)^2].$$

Thus, $\hat{Y} = a + bX$ is our guess about Y given X. The squared error is $(Y - \hat{Y})^2$. The LLSE minimizes the expected value of the squared error. Note: This is a Bayesian formulation: there is a prior.

Review: LR: Non-Bayesian or Uniform?

Observe that

$$\frac{1}{N}\sum_{n=1}^{N}(Y_n - a - bX_n)^2 = E[(Y - a - bX)^2]$$

where one assumes that

$$(X, Y) = (X_n, Y_n), \text{ w.p. } \frac{1}{N} \text{ for } n = 1, \ldots, N.$$

That is, the non-Bayesian LR is equivalent to the Bayesian LLSE that assumes that (X, Y) is uniform on the set of observed samples.
Thus, we can study the two cases LR and LLSE in one shot.
However, the interpretations are different!

Review: LLSE

Theorem
Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$. Then,

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

Non-Bayesian setting:

Review: LLSE

Theorem
Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$. Then,

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Non-Bayesian setting:

$$E[X] =$$

Review: LLSE

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Non-Bayesian setting:

$$E[X] = \frac{1}{N}\sum_{n=1}^{N} X_n; \qquad E[Y] =$$

Review: LLSE

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$$E[X] = \frac{1}{N}\sum_{n=1}^{N} X_n; \qquad E[Y] = \frac{1}{N}\sum_{n=1}^{N} Y_n$$

$$Var[X] = E[X^2] - (E[X])^2 =$$

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$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

Review: LLSE

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Consider two RVs X, Y with a given distribution $Pr[X = x, Y = y]$. Then,

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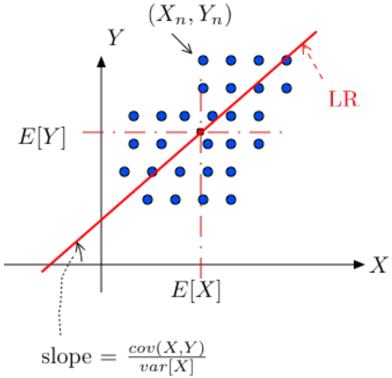
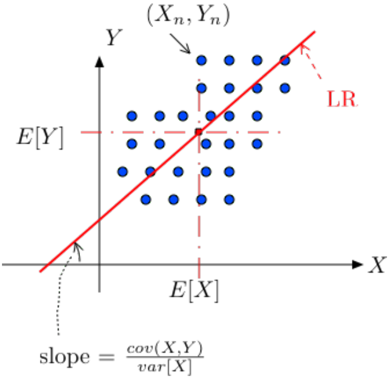
$$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{N}\sum_{n=1}^{N}(X_n)^2 - (\frac{1}{N}\sum_{n=1}^{N}(X_n))^2$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{N}\sum_{n=1}^{N}(X_nY_n) - (\frac{1}{N}\sum_{n=1}^{N} X_n)(\frac{1}{N}\sum_{n=1}^{N} Y_n)$$

LR: Illustration



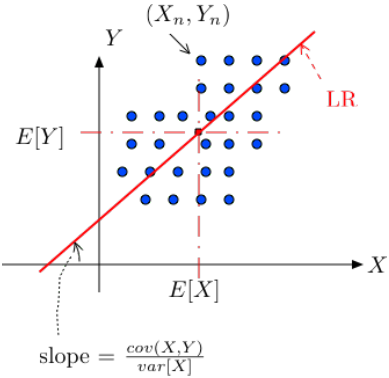
LR: Illustration



Note that

- the LR line goes through $(E[X], E[Y])$

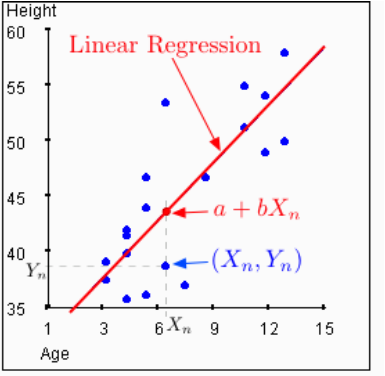
LR: Illustration



Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

Linear Regression: Examples



Example: "Removing noise or de-noising"

Y: temp. in a room (quantity of interest)

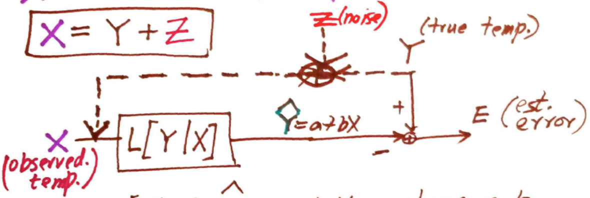
$[Y = \mathcal{N}(\mu_Y, \sigma_Y^2)]$

Z: thermal noise of temp. sensor

$[Z = \mathcal{N}(0, \sigma_Z^2)]$

X: observed (noisy) temp. measurement @ sensor

$$\boxed{X = Y + Z}$$



$L[Y|X] = \hat{Y} = a + bX$, where a, b chosen to min $\mathbb{E}\left[\begin{pmatrix}\text{est.}\\ \text{error}\end{pmatrix}^2\right] = \mathbb{E}[(Y - \hat{Y})^2]$

$$\underline{\text{LLSE:}} \quad \boxed{\hat{Y} = \mathbb{E}[Y] + \frac{\text{cov}(X,Y)}{\text{var}(X)}[X - \mathbb{E}[X]]}$$

$\cdot\ \mathbb{E}[X] = \mathbb{E}[Y+Z] = \underbrace{\mathbb{E}[Y]}_{\mu_Y} + \underbrace{\mathbb{E}[Z]}_{0} = \mu_Y$

$\cdot\ \mathrm{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y]$

$$= \underbrace{\mathbb{E}[(Y+Z)Y]}_{\underbrace{\mathbb{E}[Y^2]+\mathbb{E}[YZ]}_{\sigma_Y^2+\mu_Y^2+\underbrace{\mathbb{E}[Y]}_{\mu_Y}\cdot\underbrace{\mathbb{E}[Z]}_{0}}} - \underbrace{\mathbb{E}[Y+Z]}_{\mu_Y}\cdot\underbrace{\mathbb{E}[Y]}_{\mu_Y}$$

$$\Rightarrow \mathrm{cov}(X,Y) = \sigma_Y^2 + \mu_Y^2 - \mu_Y^2 = \boxed{\sigma_Y^2}$$

$\cdot\ \mathrm{var}(X) = \underbrace{\mathrm{var}(Y)}_{\sigma_Y^2} + \underbrace{\mathrm{var}(Z)}_{\sigma_Z^2} = \boxed{\sigma_Y^2+\sigma_Z^2}$

$$\boxed{\hat{Y} = L[Y|X] = \mu_Y + \left(\frac{\sigma_Y^2}{\sigma_Y^2+\sigma_Z^2}\right)(X-\mu_Y)}$$

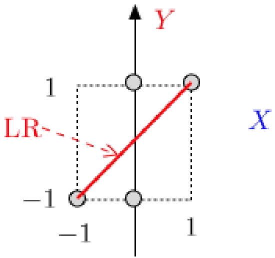
Remarks: (1) If $\sigma_Z^2 \approx 0$ (no noise) $\Rightarrow \hat{Y} \cong X$ ("believe the obs.")

(2) If $\sigma_Z^2 >> \sigma_Y^2$ (v. noisy) $\Rightarrow \hat{Y} \cong \mu_Y = \mathbb{E}[Y]$ ("believe the model & not the obs. data")

(3) If $\mu_Y = 70^0F$, $\sigma_Y^2 = 5$, $\sigma_Z^2 = 2$

$$\hat{Y} = 70 + \tfrac{5}{7}(X-70)$$

Linear Regression: Example 2



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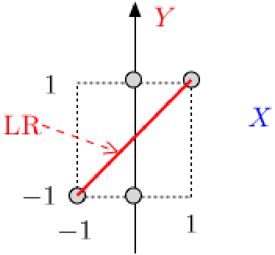
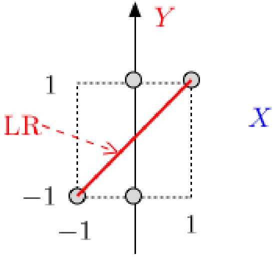


We find:

$$E[X] =$$

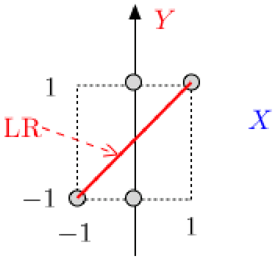
Linear Regression: Example 2



We find:

$E[X] = 0;$

Linear Regression: Example 2



We find:

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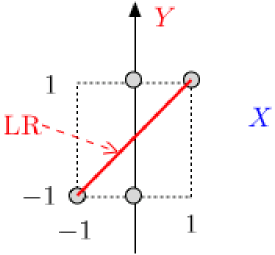

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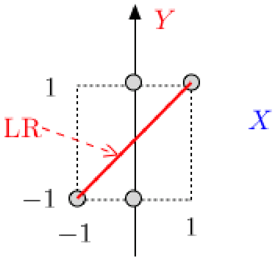

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] =$$

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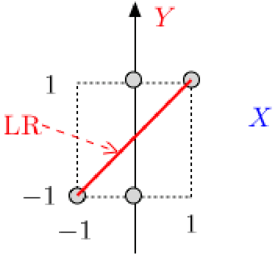

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$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2;$$

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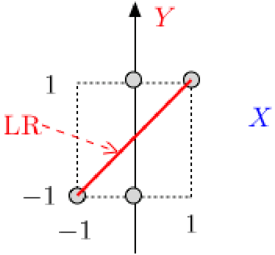

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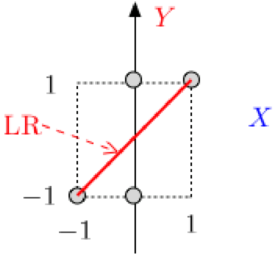

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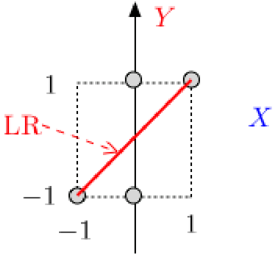

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$$var[X] = E[X^2] - E[X]^2 =$$

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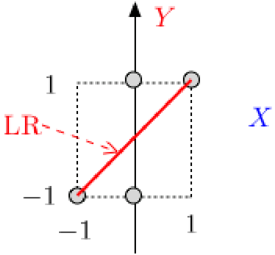

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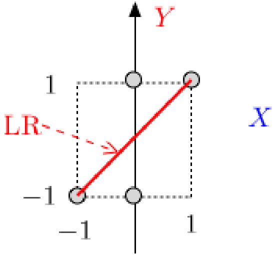

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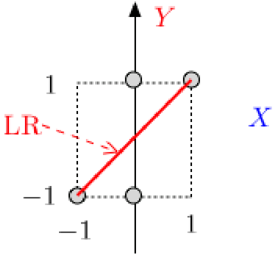

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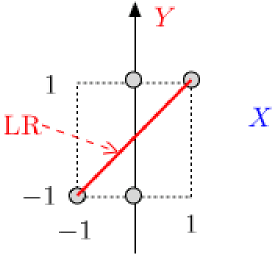

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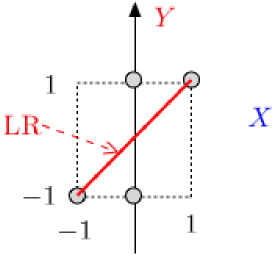
Linear Regression: Example 2



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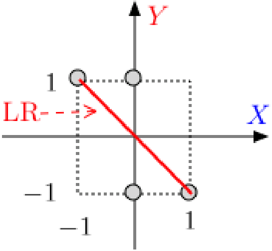
$$\text{LR: } \hat{Y} = E[Y] + \frac{cov(X, Y)}{var[X]}(X - E[X]) = X.$$

Linear Regression: Example 3



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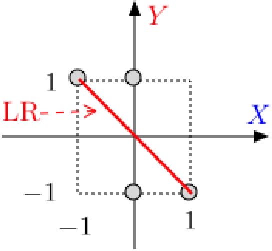

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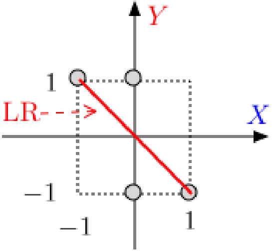

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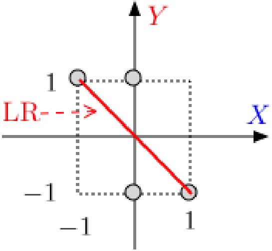
Linear Regression: Example 3



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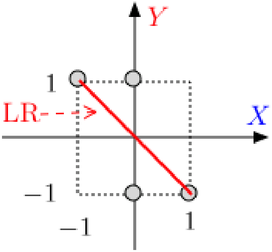

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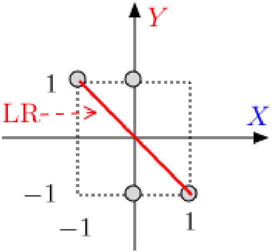
Linear Regression: Example 3



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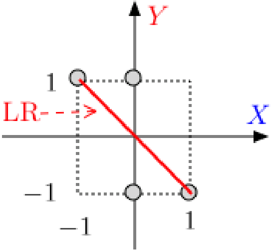
Linear Regression: Example 3



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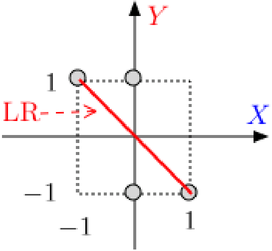

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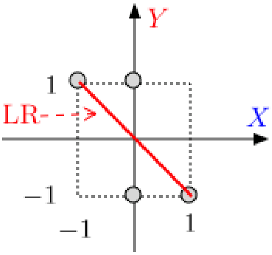

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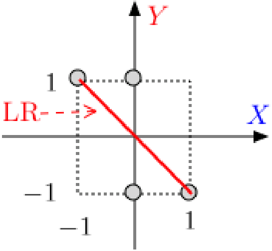
$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$

Linear Regression: Example 3



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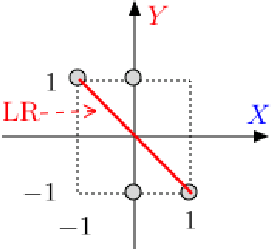
Linear Regression: Example 3



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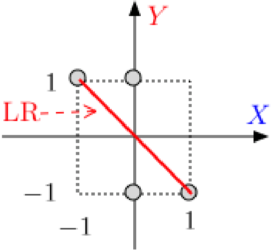

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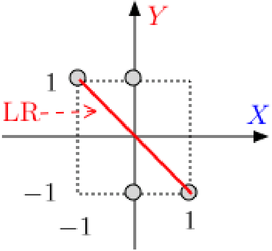

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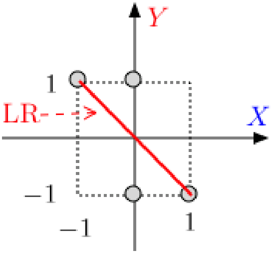

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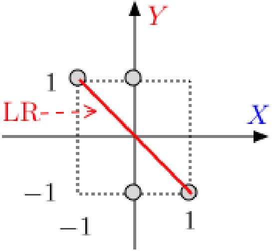

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$$\begin{aligned} E[|Y - L[Y|X]|^2] &= E[(Y - E[Y] - (cov(X,Y)/var(X))(X - E[X]))^2] \\ &= E[(Y - E[Y])^2] - 2(cov(X,Y)/var(X))E[(Y - E[Y])(X - E[X])] \\ &\quad + (cov(X,Y)/var(X))^2 E[(X - E[X])^2 \\ &= var(Y) - \frac{cov(X,Y)^2}{var(X)}. \end{aligned}$$

Without observations, the estimate is $E[Y] = 0$. The error is $var(Y)$. Observing X reduces the error.

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This is called the Conditional Expectation (CE).

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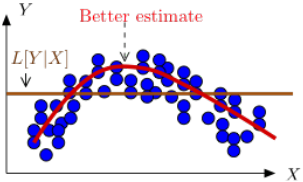

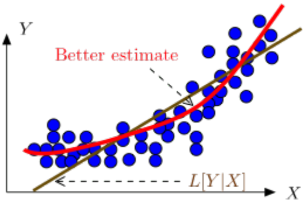

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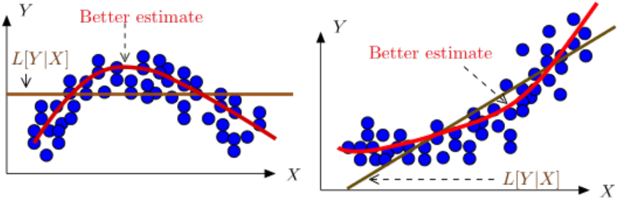

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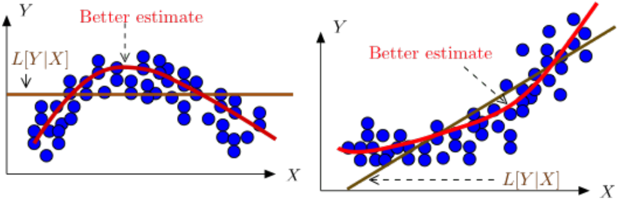

Our goal: explore estimates $\hat{Y} = g(X)$ for nonlinear functions $g(\cdot)$.

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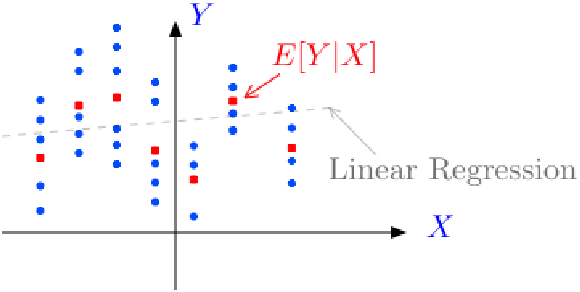

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