## CS70: Lecture 36.

Markov Chains

1. Markov Process: Motivation, Definition
2. Examples
3. Invariant Distribution of Markov Chains: Balance Equations

## From Random Variables to Random Processes

What is a random **process?**

$\Rightarrow$ Probabilistic description for a **sequence** of Random Variables $\Rightarrow$ usually associated with **time**.

*Example 1*: No. of students in my Office Hours (OH) at time $t$ (5-minute intervals)

*Example 2*: No. of dollars in my wallet at the end of a day

$X_{11/29/17} = \$17$

$X_{11/30/17} = \$7$ with probability 0.5 and $= \$13$ with probability 0.5

*Example 3*: No. of students enrolled in CS70:

Sept. 1: 800; Oct. 1: 850; Nov. 1: 750; Dec. 1: 737;

## Random Process

In general, one can describe a **random process** by describing the **joint distribution** of $(X_{t_1}, X_{t_2}, \ldots, X_{t_i})$ $\forall i \Rightarrow$ *not tractable* .

Markov Process: We make the simplifying assumption:

"Given the present, the future is decoupled from the past."

*Example*: Suppose you need to get to an 8 a.m. class, and you need to take a 7:30 a.m. bus from near your house to make it on time to class.

*Pr*[You get to your 8 a.m. class on time | You catch the 7:30 bus, You wake up at 6 a.m., You eat breakfast at 7 a.m.]

$=$ *Pr* [You get to your 8 a.m. class on time | You catch the 7:30 bus].

This is an example of the Markov property:

$P[X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots]$

$= P[X_{n+1} = x_{n+1} | X_n = x_n]$

## Example: My Office Hours (OH)

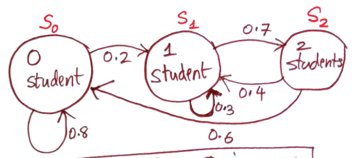
- When nobody is in my OH at time n, then at time (n+1), there will be either 1 student w.p. 0.2 or 0 student w.p. 0.8
- When 1 person is in my OH at time n, then at time (n+1), there will be either 1 student w.p. 0.3 or 2 students w.p. 0.7
- When 2 people are in my OH at time n, then at time (n+1), there will be either 0 student w.p. 0.6 or 1 student w.p. 0.4

Questions of interest:

1. How many students do I have in my OH on average?
2. If I start my OH at time 0, with 0 students, what is the probability that I have 2 students in my OH at time 10?

These questions require the study of **Markov Chains**!

## State Transition Diagram and Matrix




State Transition Diagram

State Sequence: $X_0, X_1, X_2, \ldots$ ,
where $X_i \in \{S_0, S_1, S_2\}$ $\forall i = 1, 2, \ldots$

| From \ To | $S_0$ | $S_1$ | $S_2$ |
|---|---|---|---|
| $S_0$ | 0.8 | 0.2 | 0 |
| $S_1$ | 0 | 0.3 | 0.7 |
| $S_2$ | 0.6 | 0.4 | 0 |

- Row-Sums $\equiv 1$ $\forall$ Rows
- $P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}$

state-transition matrix

$$P = \begin{bmatrix} P(0,0) & P(0,1) & P(0,2) \\ P(1,0) & P(1,1) & P(1,2) \\ P(2,0) & P(2,1) & P(2,2) \end{bmatrix}$$

$Pr[S_2 \to S_1]$

$$P(i,j) \equiv Pr[S_i \to S_j]$$

$$\underline{\pi_0} = [\pi_0(0) \quad \pi_0(1) \quad \pi_0(2)]$$

Row-vector

$$= [P(X_0 = S_0) \quad P(X_0 = S_1) \quad P(X_0 = S_2)]$$

$$\underline{\pi_1} = [P(X_1 = S_0) \quad P(X_1 = S_1) \quad P(X_1 = S_2)]$$

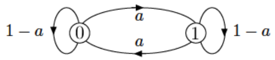
$$\vdots$$

$$\underline{\pi_n} = [P(X_n = S_0) \quad P(X_n = S_1) \quad P(X_n = S_2)]$$

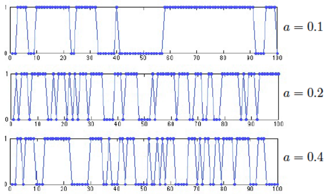
Q) How to find $\pi_n$ from $\pi_0$ ?

## Example: Two-State Markov Chain

Here is a symmetric two-state Markov chain. It describes a random motion in $\{0, 1\}$. Here, $a$ is the probability that the state changes in the next step.

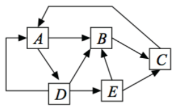



Let's simulate the Markov chain:

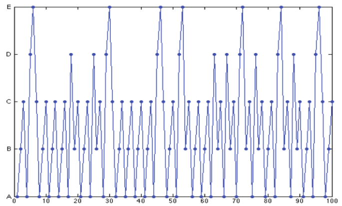



## PageRank illustration: Five-State Markov Chain

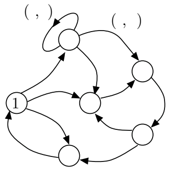
At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.




Let's simulate the Markov chain:



## Finite Markov Chain: Definition



- A finite set of states: $\mathscr{X} = \{1, 2, \ldots, K\}$
- A probability distribution $\pi_0$ on $\mathscr{X}$ : $\pi_0(i) \geq 0, \sum_i \pi_0(i) = 1$
- Transition probabilities: $P(i,j)$ for $i,j \in \mathscr{X}$

  $P(i,j) \geq 0, \forall i,j; \sum_j P(i,j) = 1, \forall i$
- $\{X_n, n \geq 0\}$ is defined so that

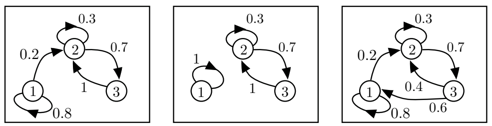
  $Pr[X_0 = i] = \pi_0(i), i \in \mathscr{X}$ (initial distribution)

  $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$.

## Irreducibility

**Definition** A Markov chain is irreducible if it can go from every state $i$ to every state $j$ (possibly in multiple steps).

**Examples:**




[A] [B] [C]

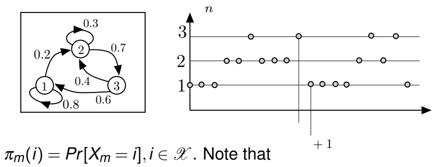
[A] is not irreducible. It cannot go from (2) to (1).

[B] is not irreducible. It cannot go from (2) to (1).

[C] is irreducible. It can go from every $i$ to every $j$.

If you consider the graph with arrows when $P(i,j) > 0$, irreducible means that there is a single connected component.

## Finding $\pi_n$: the Distribution of $X_n$



Let $\pi_m(i) = Pr[X_m = i], i \in \mathscr{X}$. Note that

$$\begin{aligned} Pr[X_{m+1} = j] &= \sum_i Pr[X_{m+1} = j, X_m = i] \\ &= \sum_i Pr[X_m = i] Pr[X_{m+1} = j \mid X_m = i] \\ &= \sum_i \pi_m(i) P(i,j). \end{aligned}$$

Hence,

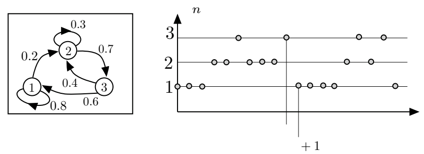
$$\pi_{m+1}(j) = \sum_i \pi_m(i) P(i,j), \forall j \in \mathscr{X}.$$

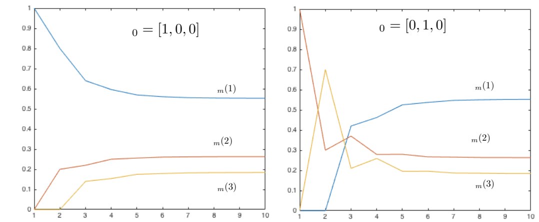
With $\pi_m, \pi_{m+1}$ as row vectors, these identities are written as $\pi_{m+1} = \pi_m P$.

Thus, $\pi_1 = \pi_0 P$, $\pi_2 = \pi_1 P = \pi_0 PP = \pi_0 P^2, \ldots$. Hence,

$$\pi_n = \pi_0 P^n, n \geq 0.$$

## OH Ex.: Finding $\pi_n$, the distribution of $X_n$





As $m$ increases, $\pi_m$ converges to a vector that does not depend on $\pi_0$.

## Balance Equations

Question: Is there some $\pi_0$ such that $\pi_m = \pi_0, \forall m$?

**Defn.** A distr. $\pi_0$ s.t. $\pi_m = \pi_0, \forall m$ is called an invariant distribution.

**Theorem** A distribution $\pi_0$ is invariant iff $\pi_0 P = \pi_0$. These equations are called the balance equations.

If $\pi_0$ is invariant, the distr. of $X_n$ is the same as that of $X_0$. Of course, this does **not** mean that nothing moves. It means that **prob. flow** leaving state $i$ = **prob. flow** entering state $i$; $\forall i \in \mathscr{X}$. That is,
**Prob. flow out = Prob. flow in for all states in the MC.**

Recall, the state transition equations from earlier slide:

$$\pi_{m+1}(j) = \sum_i \pi_m(i) P(i,j), \forall j \in \mathscr{X}.$$

The balance equations say that $\sum_j \pi(j)P(j,i) = \pi(i)$. i.e.,

$$\sum_{j \neq i} \pi(j)P(j,i) = \pi(i)(1 - P(i,i)) = \pi(i) \sum_{j \neq i} P(i,j).$$

Thus, (LHS=) $Pr[\text{enter } i] = (RHS =) Pr[\text{leave } i]$.

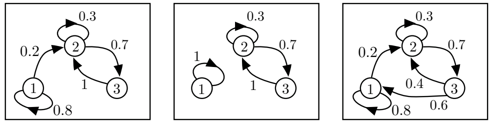
## Invariant Distribution: always exist?

**Question 1**: Does a MC *always* have an invariant distribution?

**Question 2**: If an invariant distribution exists, is it unique?

**Answer 1**: If the number of states in the MC is **finite**, then the answer to Question 1 is **yes**.

**Answer 2**: If the MC is finite and **irreducible**, then the answer to Question 2 is **yes**.

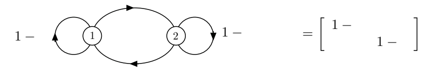


[A] [B] [C]

Proof: (EECS 126)

Other settings? (e.g. infinite chains, periodicity,...?) (EECS 126)

## Balance Equations: 2-state MC example

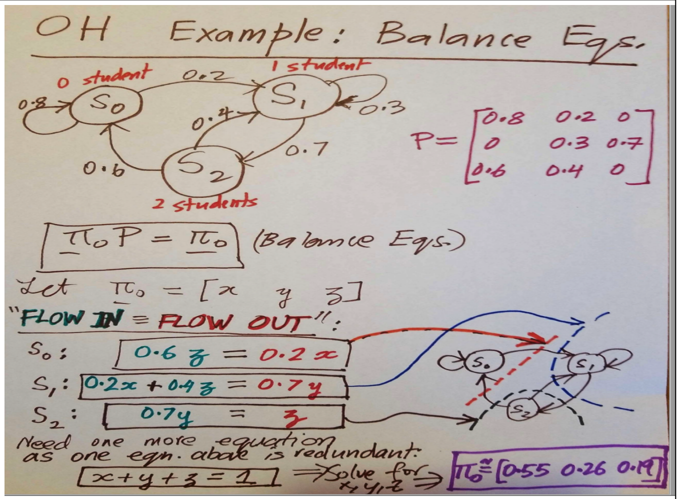



$$\pi P = \pi \quad \Leftrightarrow \quad [\pi(1), \pi(2)] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi(1), \pi(2)]$$
$$\Leftrightarrow \quad \pi(1)(1-a) + \pi(2)b = \pi(1) \text{ and } \pi(1)a + \pi(2)(1-b) = \pi(2)$$
$$\Leftrightarrow \quad \pi(1)a = \pi(2)b.$$

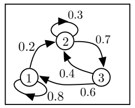
**Prob. flow leaving state 1 = Prob. flow entering state 1**

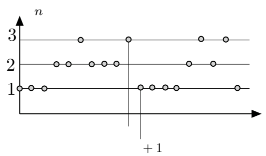
These equations are redundant! We have to add an equation: $\pi(1) + \pi(2) = 1$. Then we find

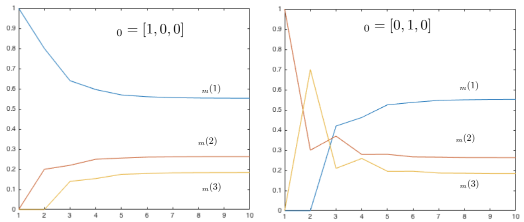
$$\pi = [\frac{b}{a+b}, \frac{a}{a+b}].$$




## Finding $\pi_n$: the Distribution of $X_n$




As $m$ increases, $\pi_m$ converges to a vector that does not depend on $\pi_0$.

## Summary

Markov Chains

1. Random Process: **sequence** of Random Variables;
2. Markov Chain: $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$
3. Invariant Distribution of Markov Chain: balance equations