CS70: Lecture 36.

Markov Chains

1. Markov Process: Motivation, Definition
2. Examples
3. Invariant Distribution of Markov Chains: Balance Equations

From Random Variables to Random Processes

What is a random **process?**

$\Rightarrow$ Probabilistic description for a **sequence** of Random Variables $\Rightarrow$ usually associated with **time**.

Example 1: No. of students in my Office Hours (OH) at time t (5-minute intervals)

Example 2: No. of dollars in my wallet at the end of a day

$X_{11/29/17} = \$17$

$X_{11/30/17} = \$7$ with probability 0.5 and $= \$13$ with probability 0.5

Example 3: No. of students enrolled in CS70:

Sept. 1: 800;   Oct. 1: 850;   Nov. 1: 750;   Dec. 1: 737;

Random Process

In general, one can describe a **random process** by describing the **joint distribution** of $(X_{t_1}, X_{t_2}, \ldots, X_{t_i})$ $\forall i \Rightarrow$ *not tractable* .

Markov Process: We make the simplifying assumption:

"Given the present, the future is decoupled from the past."

Example: Suppose you need to get to an 8 a.m. class, and you need to take a 7:30 a.m. bus from near your house to make it on time to class.

Pr[You get to your 8 a.m. class on time | You catch the 7:30 bus, You wake up at 6 a.m., You eat breakfast at 7 a.m.]

$= Pr$ [You get to your 8 a.m. class on time | You catch the 7:30 bus].

This is an example of the Markov property:
$P[X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots]$

$= P[X_{n+1} = x_{n+1} | X_n = x_n]$

Example: My Office Hours (OH)

- When nobody is in my OH at time n, then at time (n+1), there will be either 1 student w.p. 0.2 or 0 student w.p. 0.8
- When 1 person is in my OH at time n, then at time (n+1), there will be either 1 student w.p. 0.3 or 2 students w.p. 0.7
- When 2 people are in my OH at time n, then at time (n+1), there will be either 0 student w.p. 0.6 or 1 student w.p. 0.4

Questions of interest:

1. How many students do I have in my OH on average?
2. If I start my OH at time 0, with 0 students, what is the probability that I have 2 students in my OH at time 10?

These questions require the study of **Markov Chains**!

State Transition Diagram and Matrix
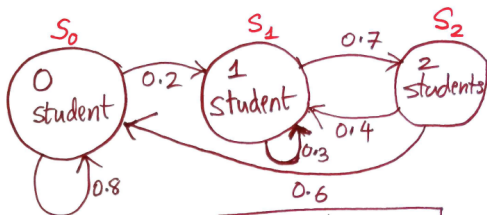
State Transition Diagram

State Sequence: $X_0, X_1, X_2, \ldots$,

where $X_i \in \{S_0, S_1, S_2\}$ $\forall i = 1, 2, \ldots$

To From	S_0	S_1	S_2
S_0	0.8	0.2	0
S_1	0	0.3	0.7
S_2	0.6	0.4	0

• Row-Sums ≡ 1 ∀ Rows

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

state-transition matrix

$$P = \begin{bmatrix} P(0,0) & P(0,1) & P(0,2) \\ P(1,0) & P(1,1) & P(1,2) \\ P(2,0) & \boxed{P(2,1)} & P(2,2) \end{bmatrix}$$

$$Pr\left[S_2 \to S_1\right]$$

$$P(i,j) \equiv Pr\left[S_i \longrightarrow S_j\right]$$

$$\underline{\pi_0} = \begin{bmatrix} \pi_0(0) & \pi_0(1) & \pi_0(2) \end{bmatrix}$$

$$\underset{\text{Row-vector}}{\uparrow}$$

$$= \begin{bmatrix} P(X_0=S_0) & P(X_0=S_1) & P(X_0=S_2) \end{bmatrix}$$

$$\underline{\pi_1} = \begin{bmatrix} P(X_1=S_0) & P(X_1=S_1) & P(X_1=S_2) \end{bmatrix}$$

$$\vdots$$

$$\underline{\pi_n} = \begin{bmatrix} P(X_n=S_0) & P(X_n=S_1) & P(X_n=S_2) \end{bmatrix}$$

Q) How to find π_n from π_0 ?

Example: Two-State Markov Chain

Here is a symmetric two-state Markov chain. It describes a random motion in $\{0, 1\}$. Here, a is the probability that the state changes in the next step.
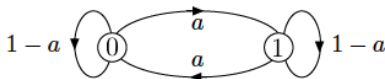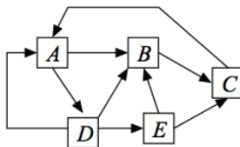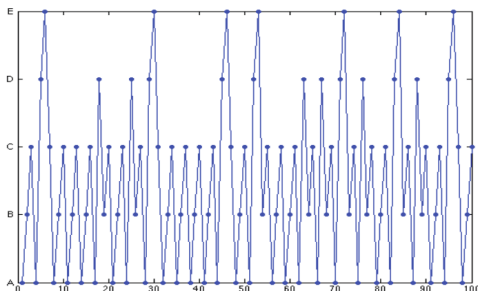

Let's simulate the Markov chain:
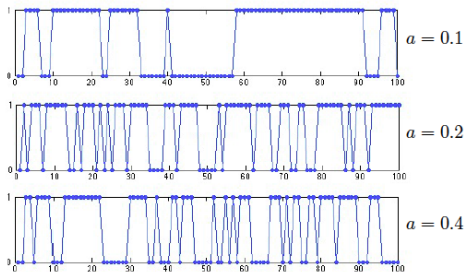
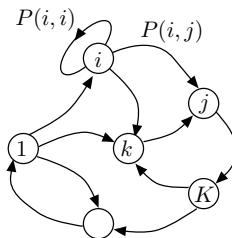# PageRank illustration: Five-State Markov Chain

At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.

Let's simulate the Markov chain:

Finite Markov Chain: Definition

- ▶ A finite set of states: $\mathscr{X} = \{1, 2, \ldots, K\}$
- ▶ A probability distribution π_0 on $\mathscr{X} : \pi_0(i) \geq 0, \sum_i \pi_0(i) = 1$
- ▶ Transition probabilities: $P(i,j)$ for $i,j \in \mathscr{X}$

  $P(i,j) \geq 0, \forall i,j; \sum_j P(i,j) = 1, \forall i$

- ▶ $\{X_n, n \geq 0\}$ is defined so that

  $Pr[X_0 = i] = \pi_0(i), i \in \mathscr{X}$ (initial distribution)

  $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}.$

Irreducibility

Definition A Markov chain is irreducible if it can go from every state *i* to every state *j* (possibly in multiple steps).
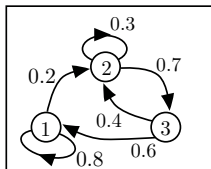
Examples:
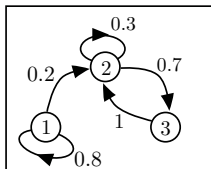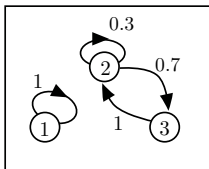
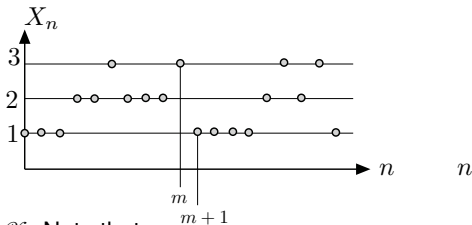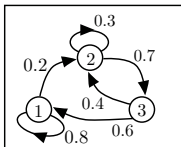[A] is not irreducible. It cannot go from (2) to (1).

[B] is not irreducible. It cannot go from (2) to (1).

[C] is irreducible. It can go from every *i* to every *j*.

If you consider the graph with arrows when $P(i,j) > 0$, irreducible means that there is a single connected component.

Finding π_n: the Distribution of X_n

Let $\pi_m(i) = Pr[X_m = i], i \in \mathcal{X}$. Note that

$$
\begin{aligned}
Pr[X_{m+1} = j] &= \sum_i Pr[X_{m+1} = j, X_m = i] \\
&= \sum_i Pr[X_m = i] Pr[X_{m+1} = j \mid X_m = i] \\
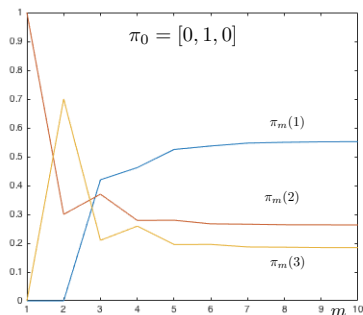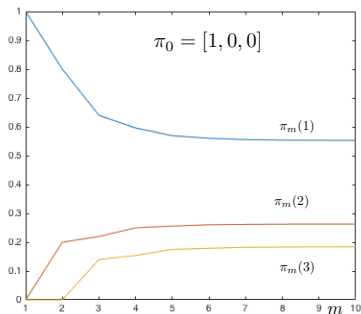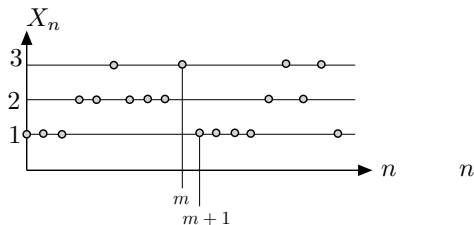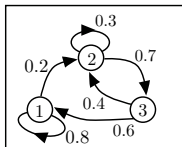&= \sum_i \pi_m(i) P(i,j).
\end{aligned}
$$

Hence,

$$
\pi_{m+1}(j) = \sum_i \pi_m(i) P(i,j), \forall j \in \mathcal{X}.
$$

With π_m, π_{m+1} as row vectors, these identities are written as $\pi_{m+1} = \pi_m P$.

Thus, $\pi_1 = \pi_0 P$, $\pi_2 = \pi_1 P = \pi_0 PP = \pi_0 P^2, \ldots$. Hence,

$$
\pi_n = \pi_0 P^n, n \geq 0.
$$

OH Ex.: Finding π_n, the distribution of X_n

As m increases, π_m converges to a vector that does not depend on π_0.

Balance Equations

Question: Is there some π_0 such that $\pi_m = \pi_0, \forall m$?

Defn. A distr. π_0 s.t. $\pi_m = \pi_0, \forall m$ is called an invariant distribution.

Theorem A distribution π_0 is invariant iff $\pi_0 P = \pi_0$. These equations are called the balance equations.

If π_0 is invariant, the distr. of X_n is the same as that of X_0. Of course, this does **not** mean that nothing moves. It means that **prob. flow** leaving state i = **prob. flow** entering state i; $\forall i \in \mathcal{X}$. That is, **Prob. flow out = Prob. flow in for all states in the MC.**

Recall, the state transition equations from earlier slide:

$$\pi_{m+1}(j) = \sum_i \pi_m(i) P(i,j), \forall j \in \mathcal{X}.$$

The balance equations say that $\sum_j \pi(j) P(j,i) = \pi(i)$. i.e.,

$$\sum_{j \neq i} \pi(j) P(j,i) = \pi(i)(1 - P(i,i)) = \pi(i) \sum_{j \neq i} P(i,j).$$

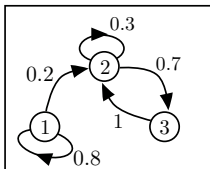Thus, (LHS=) $Pr[\text{enter } i] = (RHS =) Pr[\text{leave } i]$.

Invariant Distribution: always exist?

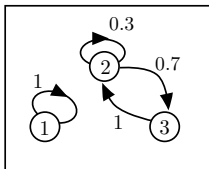**Question 1**: Does a MC *always* have an invariant distribution?

Question 2: If an invariant distribution exists, is it unique?

Answer 1: If the number of states in the MC is **finite**, then the answer to Question 1 is **yes**.
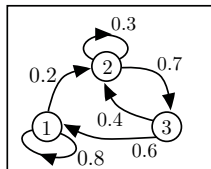
Answer 2: If the MC is finite and **irreducible**, then the answer to Question 2 is **yes**.
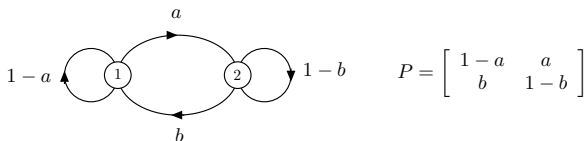

$$[A] \qquad [B] \qquad [C]$$

Proof: (EECS 126)

Other settings? (e.g. infinite chains, periodicity,...?) (EECS 126)

Balance Equations: 2-state MC example

$$\pi P = \pi \quad \Leftrightarrow \quad [\pi(1), \pi(2)] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi(1), \pi(2)]$$
$$\Leftrightarrow \quad \pi(1)(1-a) + \pi(2)b = \pi(1) \text{ and } \pi(1)a + \pi(2)(1-b) = \pi(2)$$
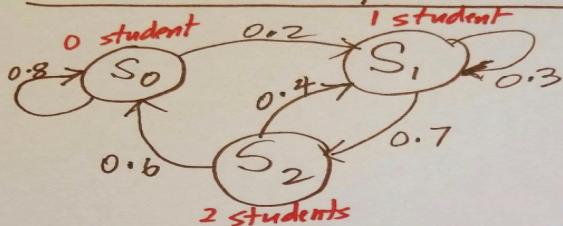$$\Leftrightarrow \quad \pi(1)a = \pi(2)b.$$

Prob. flow leaving state 1 = Prob. flow entering state 1

These equations are redundant! We have to add an equation:
$\pi(1) + \pi(2) = 1$. Then we find

$$\pi = [\frac{b}{a+b}, \frac{a}{a+b}].$$

OH Example: Balance Eqs.

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$\boxed{\underline{\pi_0} P = \underline{\pi_0}} \quad \text{(Balance Eqs.)}$$

Let $\underline{\pi_0} = [x \quad y \quad z]$

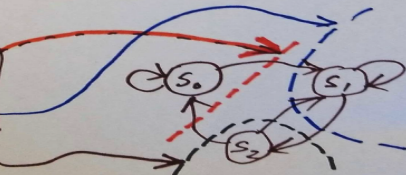**"FLOW IN ≡ FLOW OUT":**

S_0: $\boxed{0.6 z = 0.2 x}$

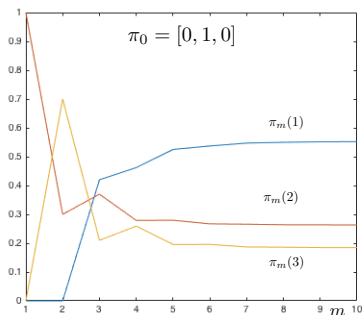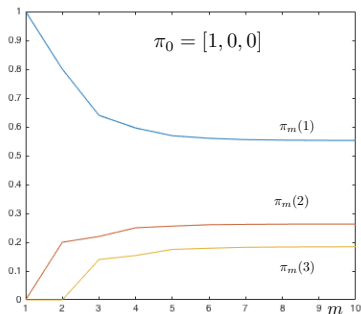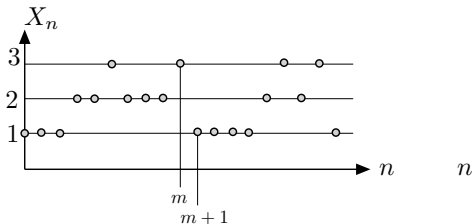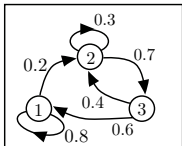S_1: $\boxed{0.2x + 0.4z = 0.7y}$

S_2: $\boxed{0.7y = z}$

Need one more equation
as one eqn. above is redundant:
$\boxed{x + y + z = 1}$ $\Rightarrow$ Solve for x, y, z $\Rightarrow$ $\boxed{\underline{\pi_0} \cong [0.55 \ 0.26 \ 0.19]}$

Finding π_n: the Distribution of X_n

As m increases, π_m converges to a vector that does not depend on π_0.

Summary

Markov Chains

1. Random Process: **sequence** of Random Variables;
2. Markov Chain: $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathcal{X}$
3. Invariant Distribution of Markov Chain: balance equations