CS70: Lecture 36.

Markov Chains

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1. Markov Process: Motivation, Definition
2. Examples
3. Invariant Distribution of Markov Chains: Balance Equations

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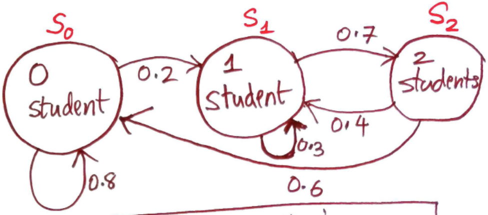
These questions require the study of **Markov Chains**!

State Transition Diagram and Matrix



State Transition Diagram

State Sequence: $X_0, X_1, X_2, \ldots$,
where $X_i \in \{S_0, S_1, S_2\}$ $\forall i = 1, 2, \ldots$

From \ To	S_0	S_1	S_2
S_0	0.8	0.2	0
S_1	0	0.3	0.7
S_2	0.6	0.4	0

- Row-Sums ≡ 1 ∀ Rows
- $$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

state-transition matrix

$$P = \begin{bmatrix} P(0,0) & P(0,1) & P(0,2) \\ P(1,0) & P(1,1) & P(1,2) \\ P(2,0) & P(2,1) & P(2,2) \end{bmatrix}$$

$P(2,1) \rightarrow Pr[S_2 \rightarrow S_1]$

$$P(i,j) \equiv Pr[S_i \longrightarrow S_j]$$

$$\underline{\pi}_0 = [\pi_0(0) \quad \pi_0(1) \quad \pi_0(2)]$$

↑ Row-vector

$$= [P(X_0 = S_0) \quad P(X_0 = S_1) \quad P(X_0 = S_2)]$$

$$\underline{\pi}_1 = [P(X_1 = S_0) \quad P(X_1 = S_1) \quad P(X_1 = S_2)]$$

$$\vdots$$

$$\underline{\pi}_n = [P(X_n = S_0) \quad P(X_n = S_1) \quad P(X_n = S_2)]$$

Q) How to find π_n from π_0?

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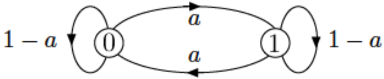

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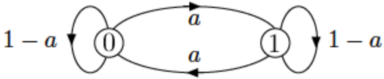

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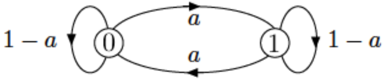

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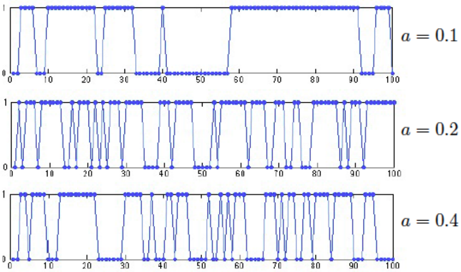
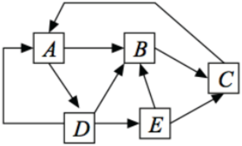


PageRank illustration: Five-State Markov Chain

At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.



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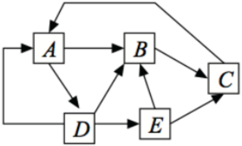

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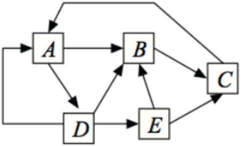

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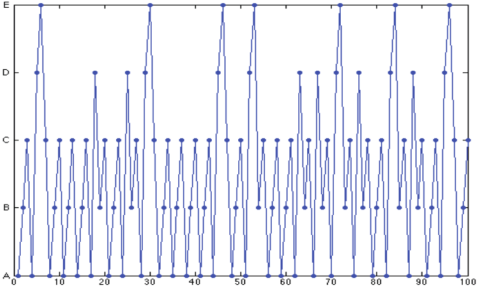

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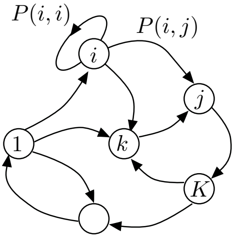

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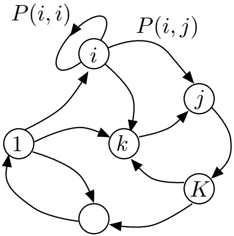

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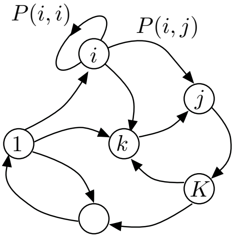

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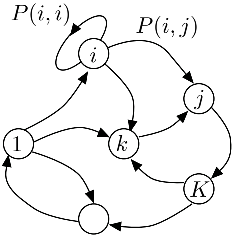

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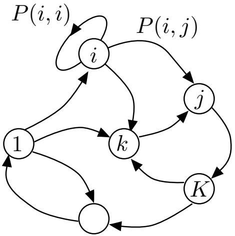

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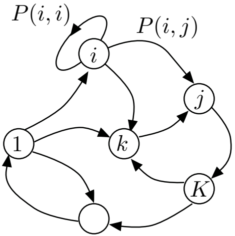

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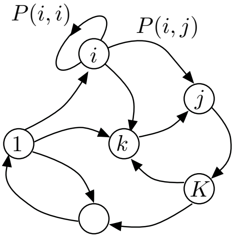

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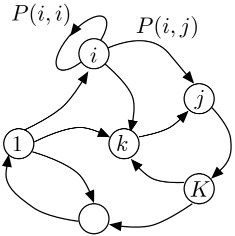

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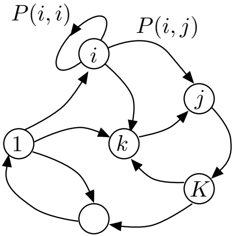

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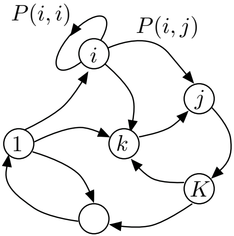

- A finite set of states: $\mathscr{X} = \{1, 2, \ldots, K\}$
- A probability distribution π_0 on $\mathscr{X}$: $\pi_0(i) \geq 0, \sum_i \pi_0(i) = 1$
- Transition probabilities: $P(i,j)$ for $i,j \in \mathscr{X}$

  $P(i,j) \geq 0, \forall i,j; \sum_j P(i,j) = 1, \forall i$
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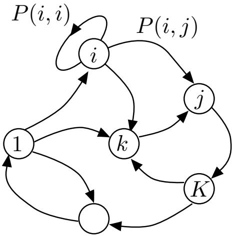

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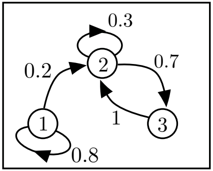
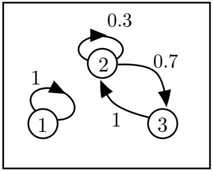
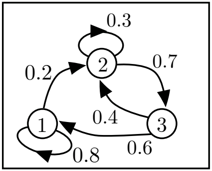


[A]



[B]



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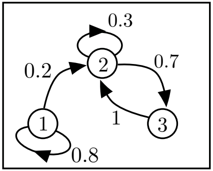

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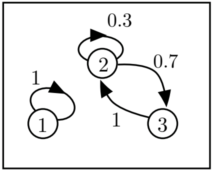

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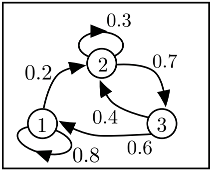

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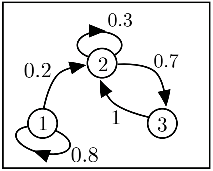

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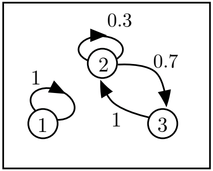

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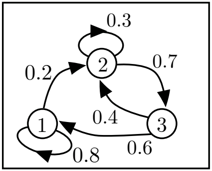

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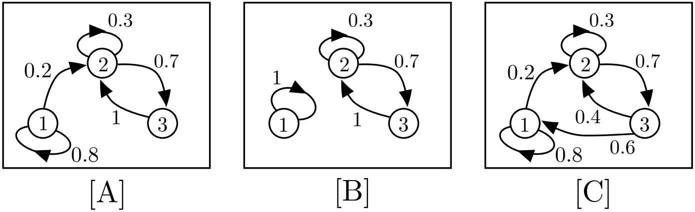

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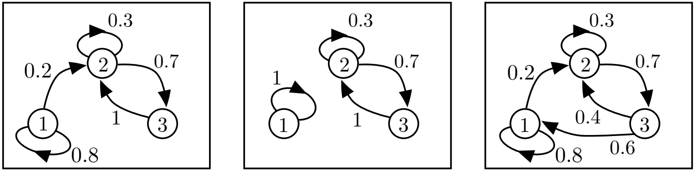

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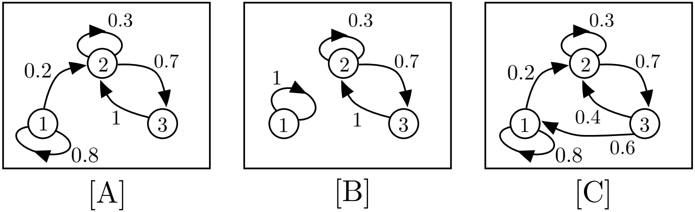

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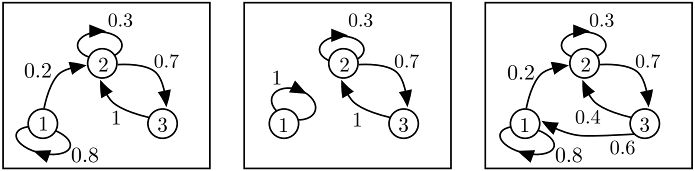

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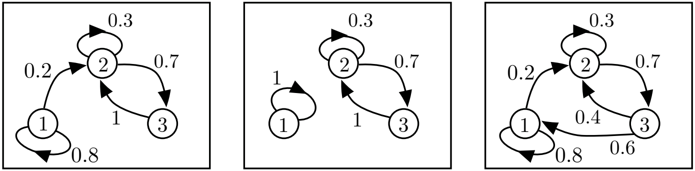

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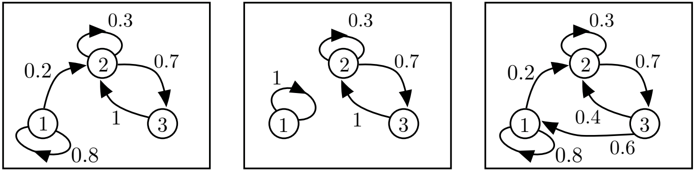

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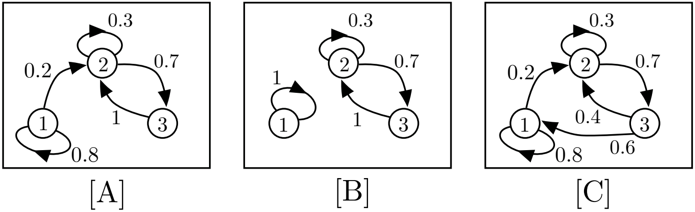

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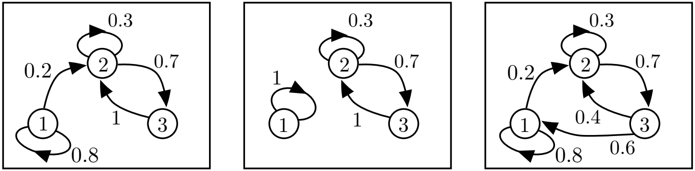

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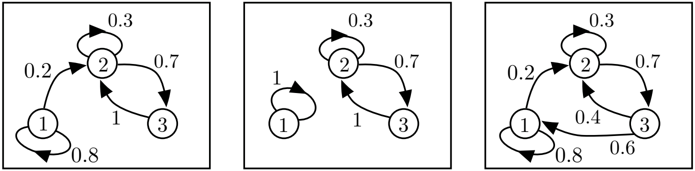

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Finding π_n: the Distribution of X_n

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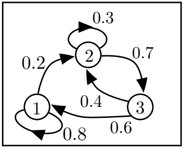

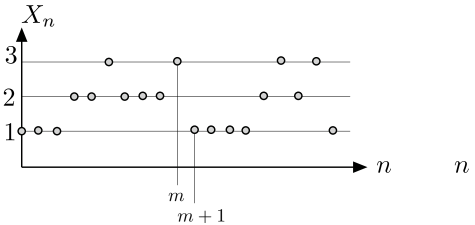

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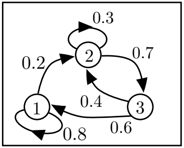

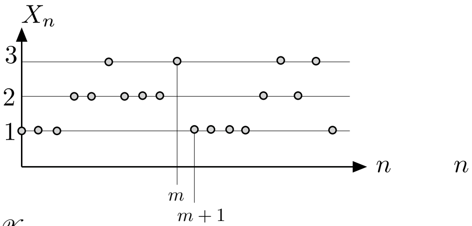

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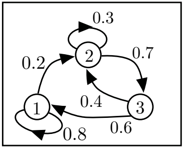

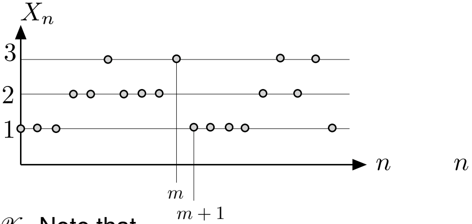

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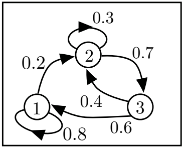

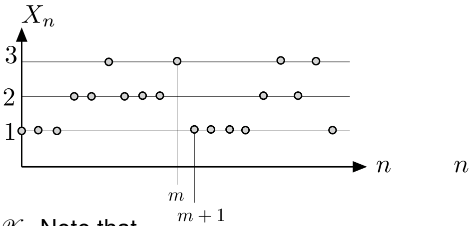

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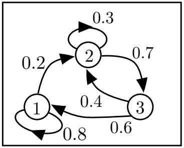

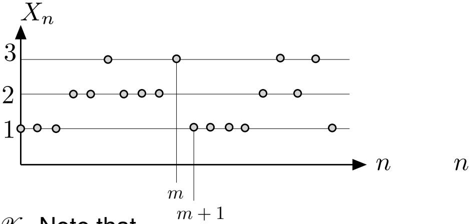

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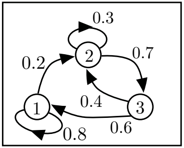

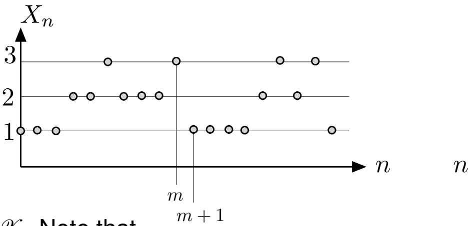

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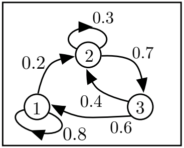

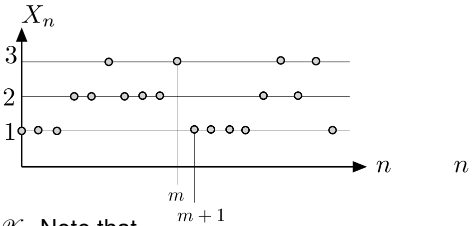

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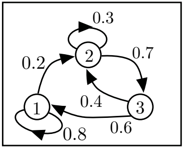

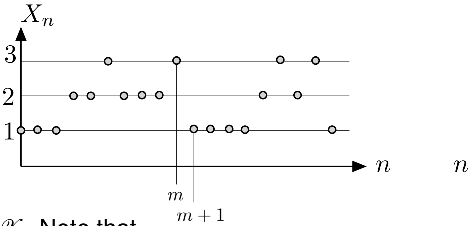

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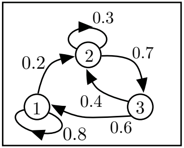

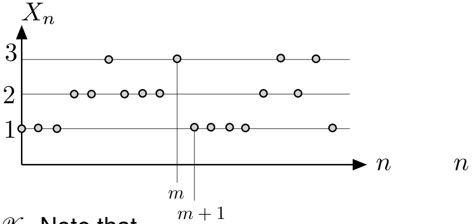

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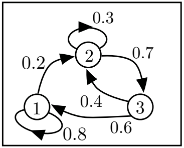

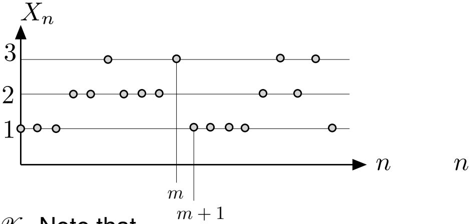

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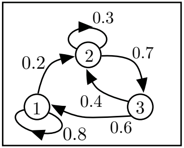

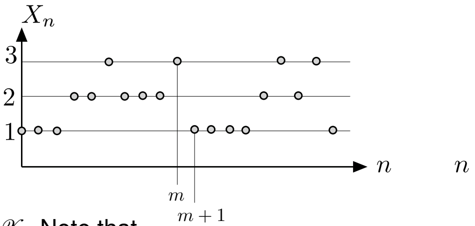

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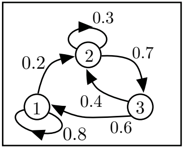

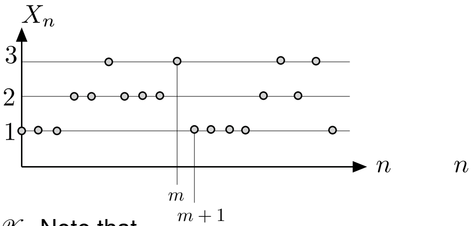

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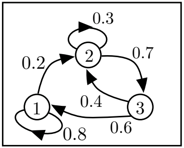
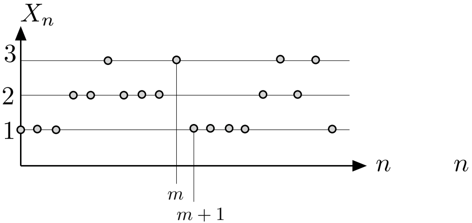
$$\pi_n = \pi_0 P^n, n \geq 0.$$

OH Ex.: Finding π_n, the distribution of X_n





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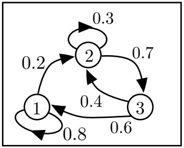

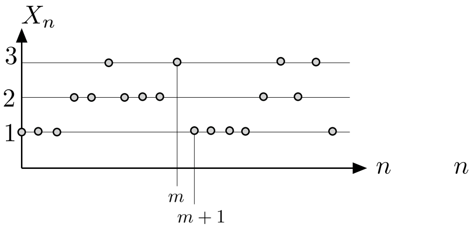

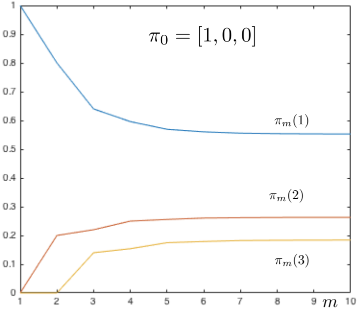

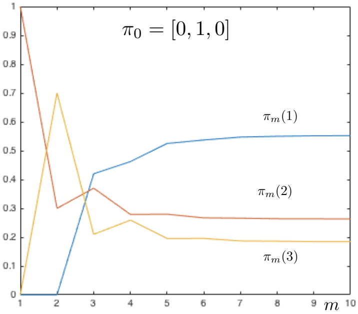

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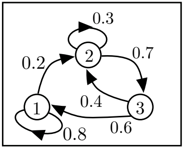

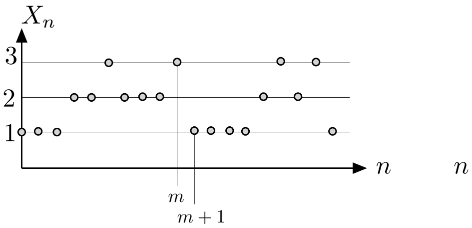

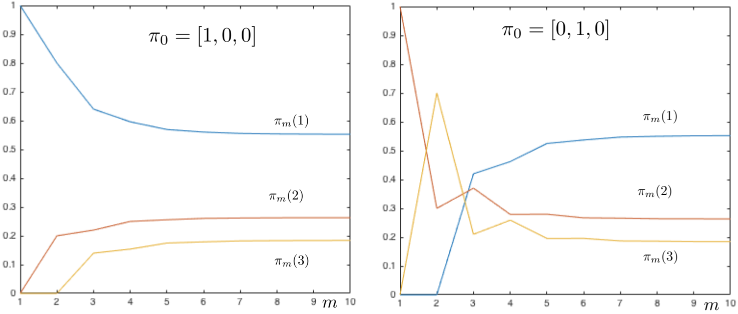

As m increases, π_m converges to a vector that does not depend on π_0.

Balance Equations

Balance Equations

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Recall, the state transition equations from earlier slide:

$$\pi_{m+1}(j) = \sum_i \pi_m(i) P(i,j), \forall j \in \mathscr{X}.$$

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Thus, (LHS=) $Pr[\text{enter } i] = (RHS =) Pr[\text{leave } i]$.

Invariant Distribution: always exist?

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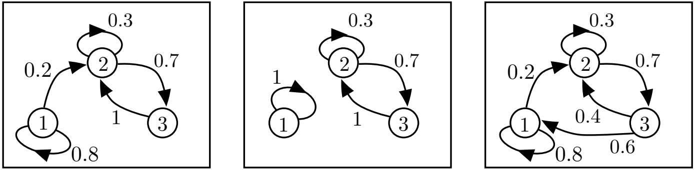

[A] [B] [C]

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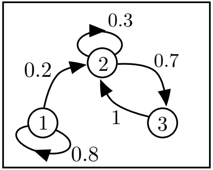
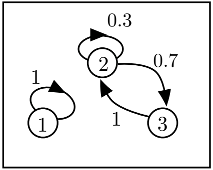


[A]



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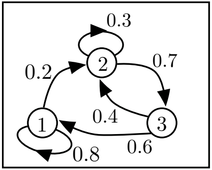

[C]

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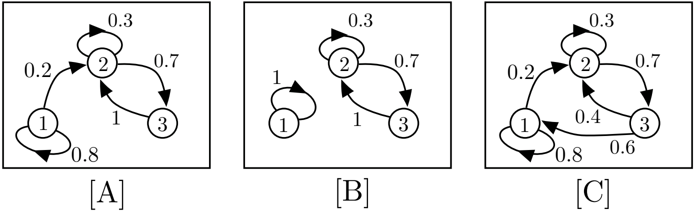

Proof: (EECS 126)

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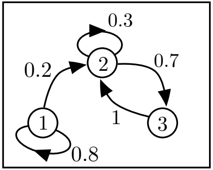

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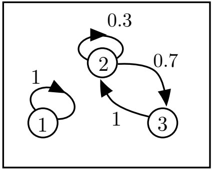

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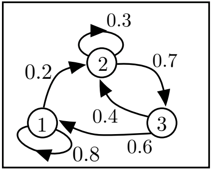

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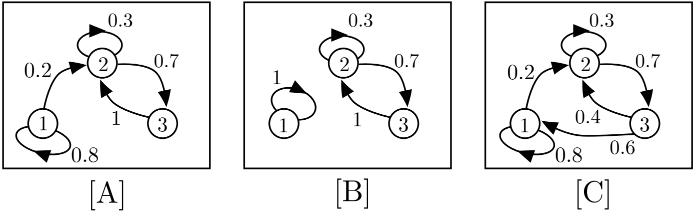

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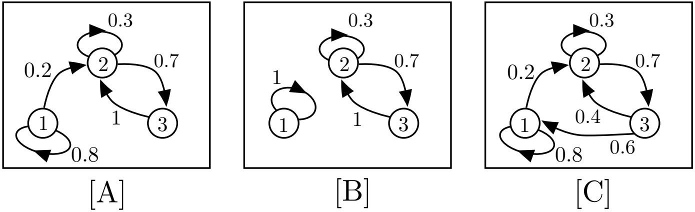

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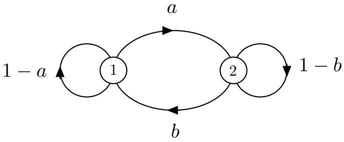

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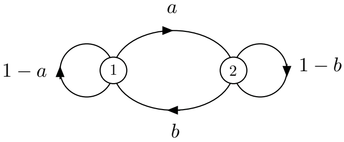

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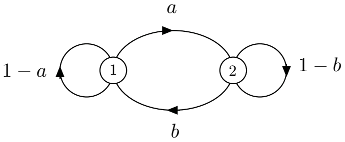

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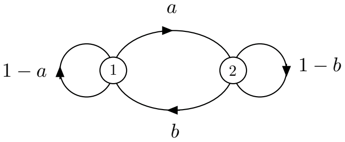

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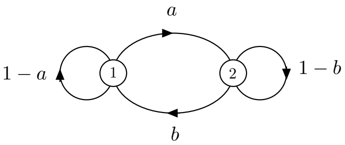

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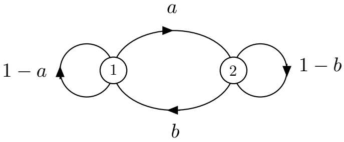

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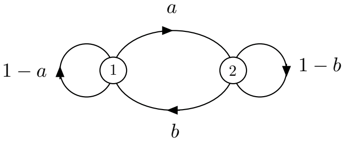

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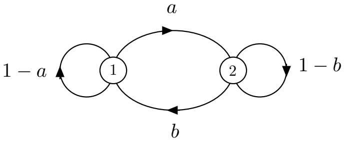

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These equations are redundant!

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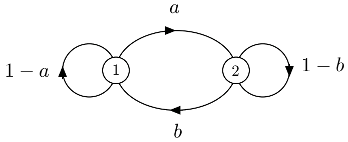

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These equations are redundant! We have to add an equation:

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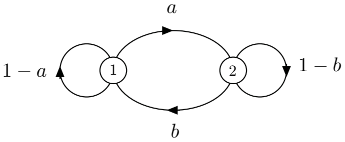

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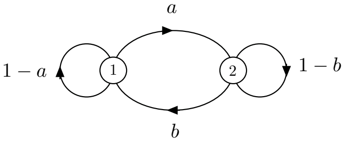
These equations are redundant! We have to add an equation:
$\pi(1) + \pi(2) = 1$.

Balance Equations: 2-state MC example



$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$\pi P = \pi \quad \Leftrightarrow \quad [\pi(1), \pi(2)] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi(1), \pi(2)]$$

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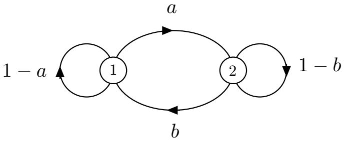

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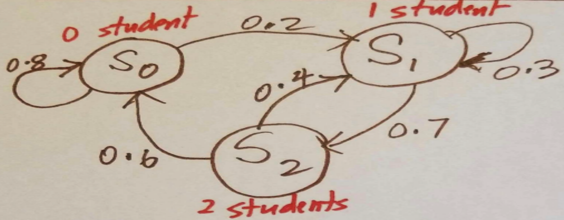
These equations are redundant! We have to add an equation: $\pi(1) + \pi(2) = 1$. Then we find

$$\pi = [\frac{b}{a+b}, \frac{a}{a+b}].$$

OH Example: Balance Eqs.



$$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$\boxed{\underline{\pi}_0 P = \underline{\pi}_0}$$ (Balance Eqs.)

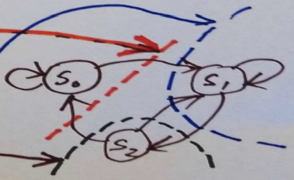
Let $\underline{\pi}_0 = [x \quad y \quad z]$

"FLOW IN = FLOW OUT":

S_0: $0.6z = 0.2x$

S_1: $0.2x + 0.4z = 0.7y$

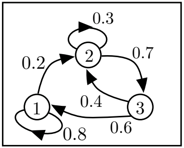
S_2: $0.7y = z$

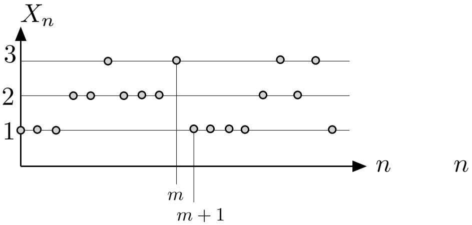


Need one more equation as one eqn. above is redundant:

$\boxed{x + y + z = 1}$ ⇒ Solve for x, y, z ⇒ $\boxed{\underline{\pi}_0 \cong [0.55 \quad 0.26 \quad 0.19]}$

Finding π_n: the Distribution of X_n





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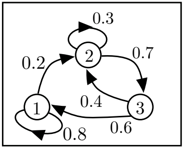

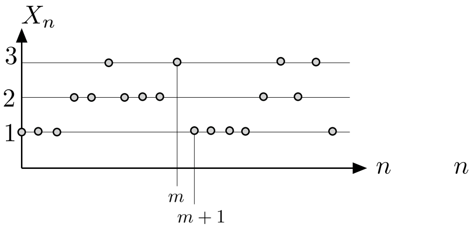

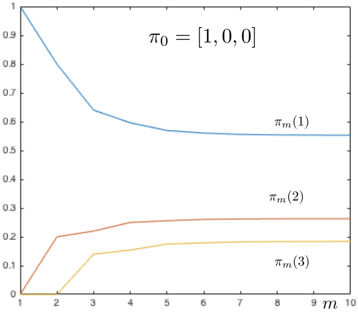

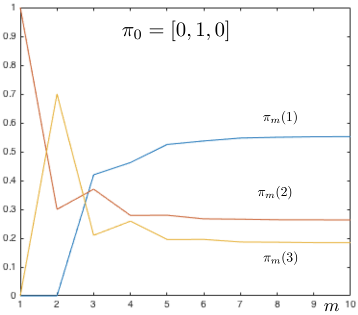

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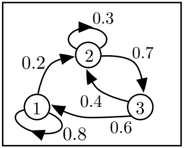

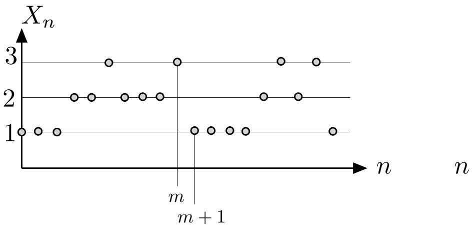

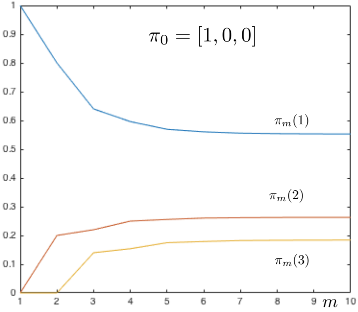

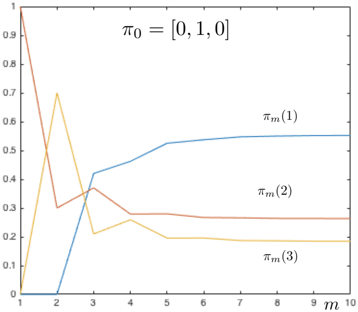

As m increases, π_m converges to a vector that does not depend on π_0.

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