CS70: Lecture 37.

Markov Chains (contd.): First Passage Time: First Step Equations

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1. Brief Recap of Markov Chains
2. First Passage Time – First Step Equations
3. Parting Thoughts

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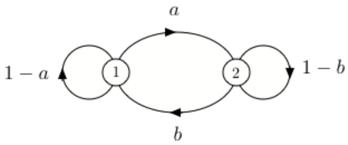
- Balance Equations: Prob. Flow In = Prob. Flow out of every state.

Balance Equations: 2-state MC example



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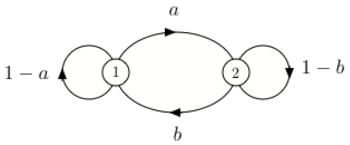

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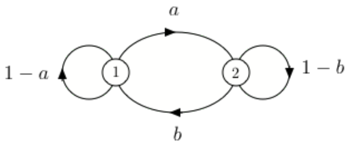

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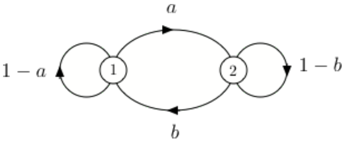

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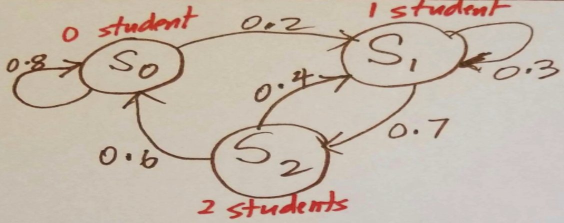
These equations are redundant! We have to add an equation: $\pi(1) + \pi(2) = 1$. Then we find

$$\pi = [\frac{b}{a+b}, \frac{a}{a+b}].$$

OH Example: Balance Eqs.



$$P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$\boxed{\underline{\pi}_0 P = \underline{\pi}_0}$$ (Balance Eqs.)

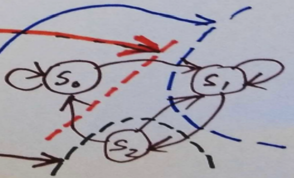
Let $\underline{\pi}_0 = [x \quad y \quad z]$

"FLOW IN = FLOW OUT":

S_0: $0.6z = 0.2x$

S_1: $0.2x + 0.4z = 0.7y$

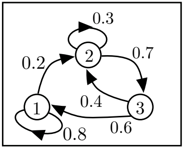
S_2: $0.7y = z$

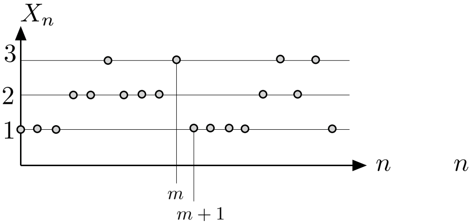


Need one more equation as one eqn. above is redundant:

$\boxed{x + y + z = 1}$ ⇒ Solve for x, y, z ⇒ $\boxed{\underline{\pi}_0 \cong [0.55 \;\; 0.26 \;\; 0.19]}$

Distribution of X_n





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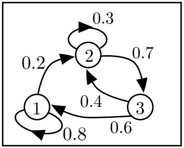

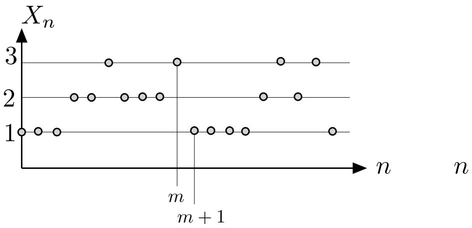

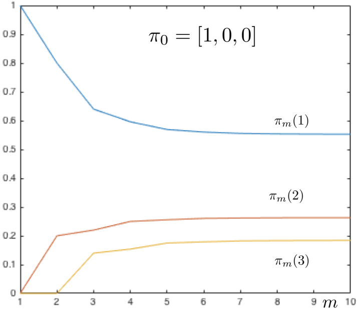

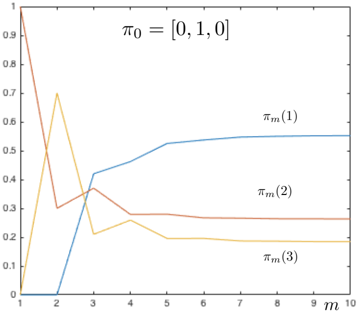

As m increases, π_m converges to a vector that does not depend on π_0.

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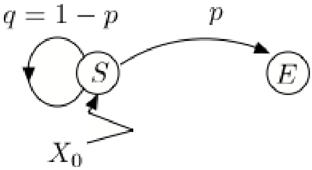
Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



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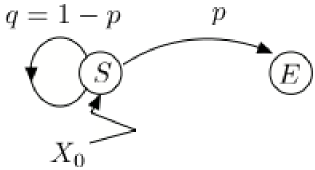

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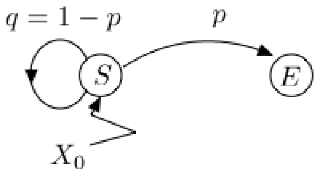

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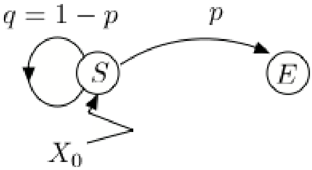

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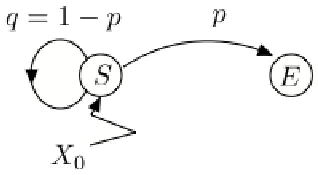

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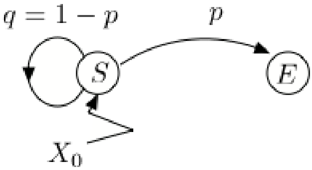

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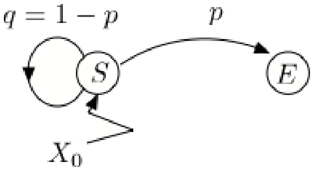

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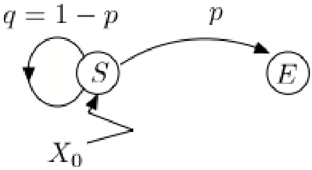

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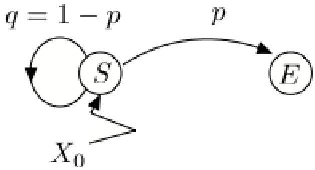

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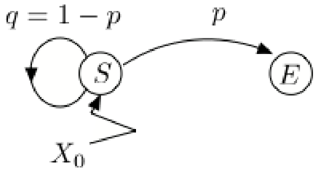

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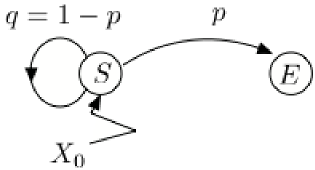

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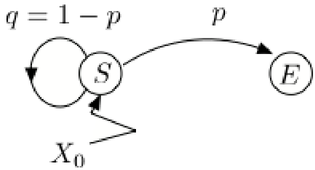

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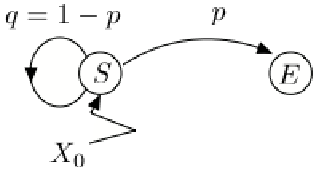

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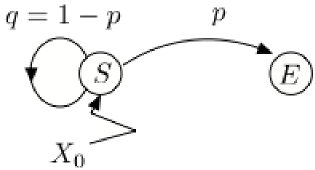

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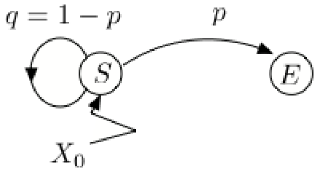

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Now, Z and N' are independent.

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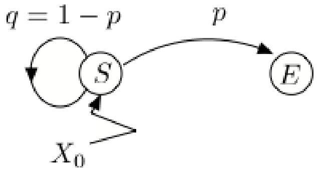

Let $\beta(S)$ be the average time until E. Then,

$$\beta(S) = 1 + (q \times \beta(S)) + (p \times 0).$$

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Let N be the random number of steps until E, starting from S.

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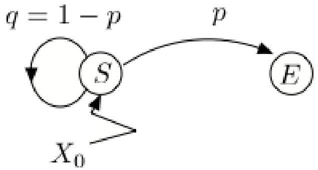

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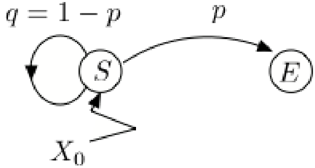

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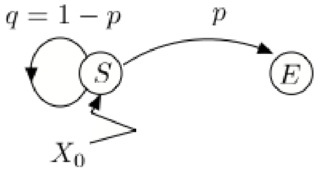

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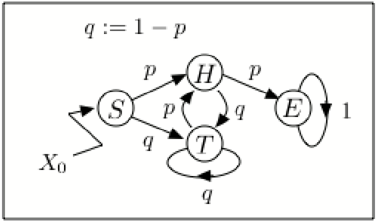

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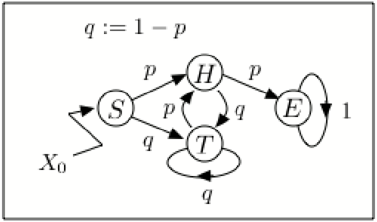

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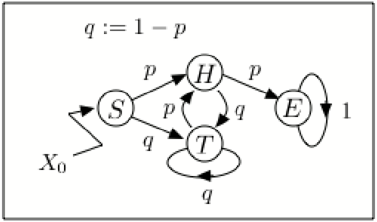

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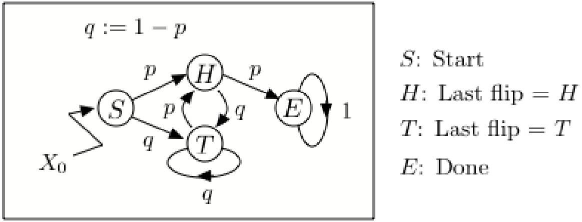

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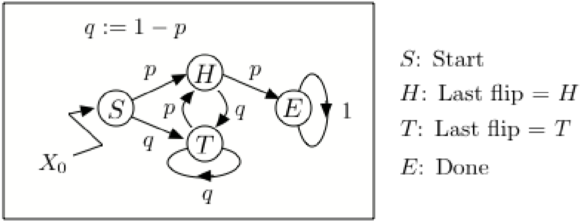

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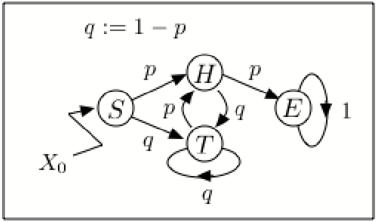

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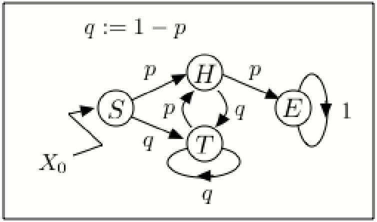

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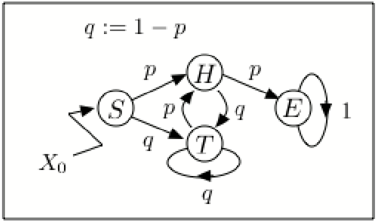

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$$\begin{aligned}\beta(S) &= 1 + p\beta(H) + q\beta(T)\\ \beta(H) &= 1 + p0 + q\beta(T)\\ \beta(T) &= 1 + p\beta(H) + q\beta(T).\end{aligned}$$

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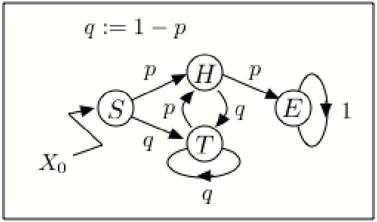

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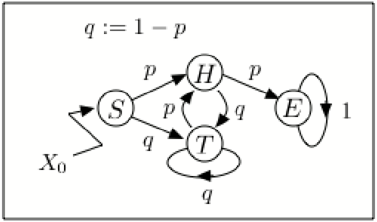

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Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$. (E.g., $\beta(S) = 6$ if $p = 1/2$.)

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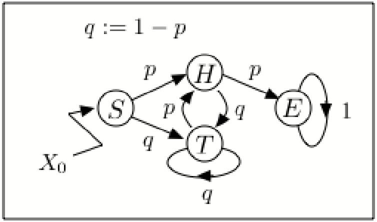

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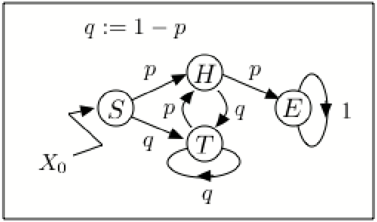

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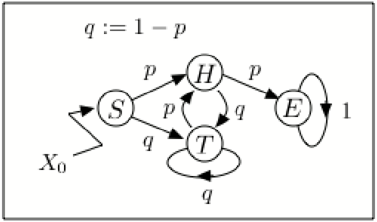
Let us justify the first step equation for $\beta(T)$.

First Passage Time - Example 2



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Let us justify the first step equation for $\beta(T)$. The others are similar.

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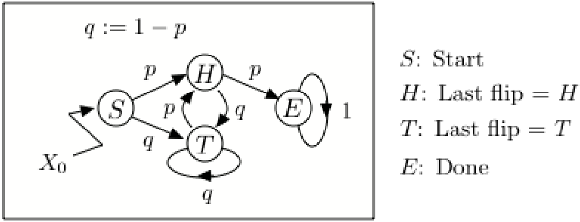

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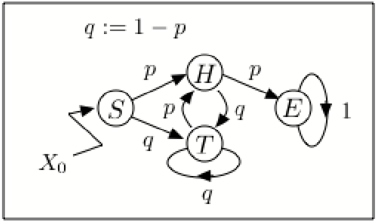

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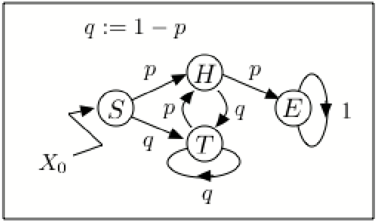

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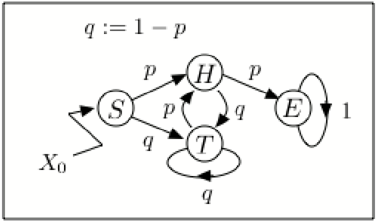

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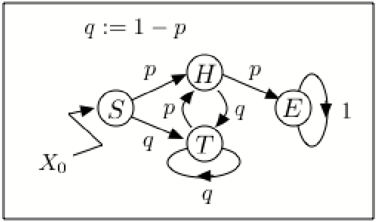

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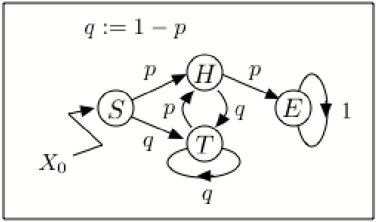

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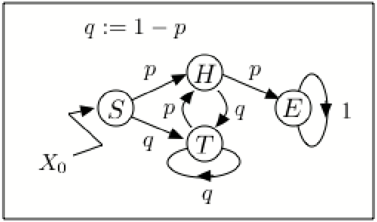

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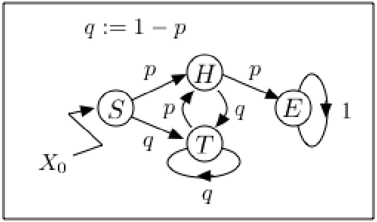

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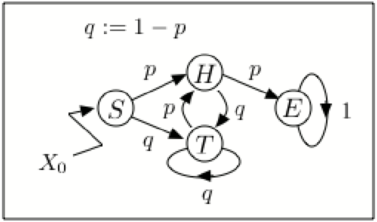

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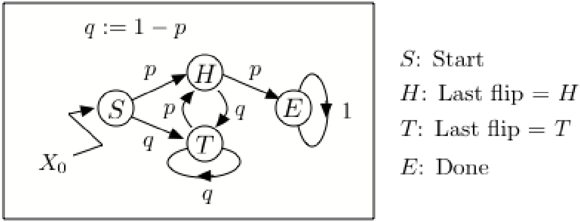

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$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

i.e.,

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First Passage Time - Example 3: Practice Exercise

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Spoiler Alert: Solution on next slide

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Spoiler Alert: Solution on next slide

(but don't look: try to do it yourself first!)

Example 3: Practice Exercise Solution

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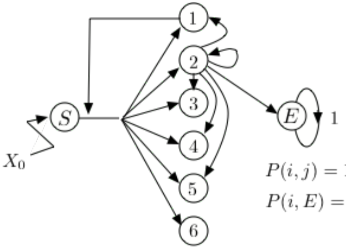

S = Start; E = Done

i = Last roll is i, not done

$P(S, j) = 1/6, j = 1, \ldots, 6$

$P(1, j) = 1/6, j = 1, \ldots, 6$

$P(i, j) = 1/6, i = 2, \ldots, 6; 8 - i \neq j \in \{1, \ldots, 6\}$

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The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

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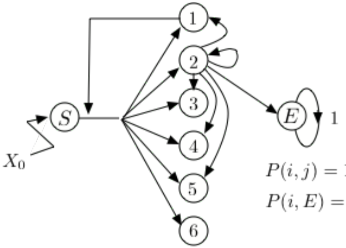

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The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

$$\beta(S) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j);$$

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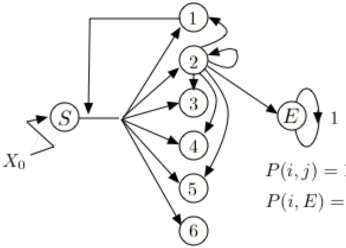

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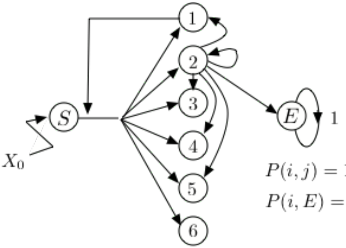

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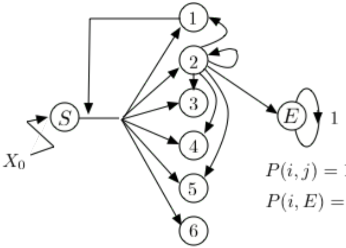

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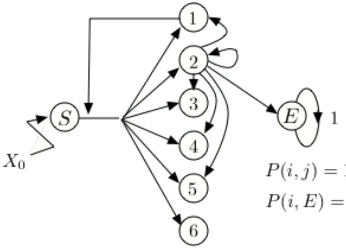

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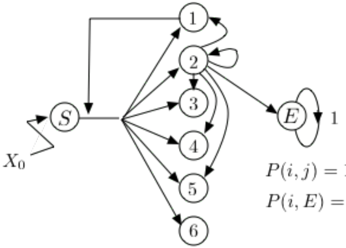

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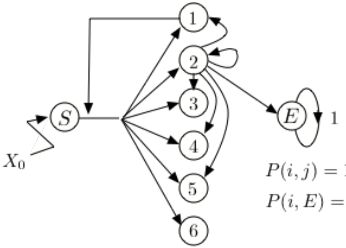

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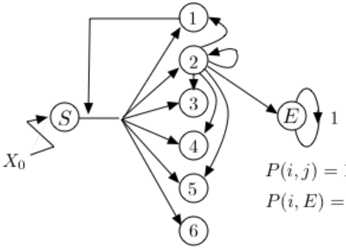

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- The Texas sharpshooter fallacy. Look at people living close to power lines. You find clusters of cancers. You will also find such clusters when looking at people eating kale.
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More on Confusing Statistics

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- Beware of statistics reported in the media!

Confirmation Bias

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Confirmation Bias: An experiment

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Surprisingly, people tend to be reinforced in their original belief, even when the evidence accumulates against it.

Being Rational: ‘Thinking, Fast and Slow’

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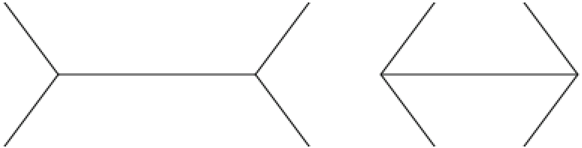

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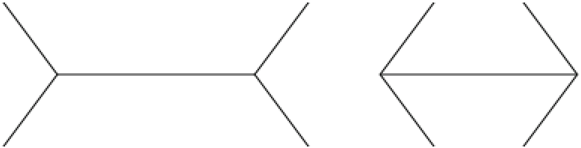

It is difficult to think clearly!

What's Next?

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GOOD LUCK IN YOUR FINAL EXAM!!!