Markov Chains (contd.): First Passage Time: First Step Equations
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1. Brief Recap of Markov Chains
2. First Passage Time – First Step Equations
3. Parting Thoughts
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Irreducible MC: every state reachable from every other state

State-transition equations:

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These equations are called the balance equations.

Finite irreducible Markov Chains have unique invariant distribution.

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Balance Equations: 2-state MC example

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These equations are redundant! We have to add an equation: $\pi(1) + \pi(2) = 1$. Then we find

$$\pi = \left[ \frac{b}{a+b}, \frac{a}{a+b} \right].$$
Example: Balance Eqs.

\[ P = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0 \end{bmatrix} \]

Let \[ \pi_0 = [x \ y \ z] \]

"Flow in = Flow out":

- \( S_0: 0.6z = 0.2x \)
- \( S_1: 0.2x + 0.4z = 0.7y \)
- \( S_2: 0.7y = z \)

Need one more equation as one eqn. above is redundant:

\[ x + y + z = 1 \]
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As $m$ increases, $\pi_m$ converges to a vector that does not depend on $\pi_0$. 

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Let’s flip a coin with \( Pr[H] = p \) until we get \( H \). How many flips, on average?

\[
q = 1 - p
\]

\[
p
\]

\[
\beta(S) = 1 + (q \times \beta(S)) + (p \times 0)
\]

(See next slide.)

Hence,

\[
p \beta(S) = 1
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so that

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\beta(S) = \frac{1}{p}
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Note: Time until \( E \) is \( G(p) \).

We have rediscovered that the mean of \( G(p) \) is \( \frac{1}{p} \).
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\[ H: \text{Last flip} = H \]
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Solving, we find
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\]

Solving, we find \( \beta(S) = 2 + 3qp^{-1} + q^{2}p^{-2} \).
Let's flip a coin with $Pr[H] = p$ until we get two consecutive $H$s. How many flips, on average? Here is a picture:

Let $\beta(i)$ be the average time from state $i$ until the MC hits state $E$. We claim that (these are called the first step equations)

\[
\begin{align*}
\beta(S) & = 1 + p\beta(H) + q\beta(T) \\
\beta(H) & = 1 + p0 + q\beta(T) \\
\beta(T) & = 1 + p\beta(H) + q\beta(T).
\end{align*}
\]

Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$. (E.g., $\beta(S) = 6$ if $p = 1/2$.)
Let us justify the first step equation for $\beta(T)$. The others are similar. Let $N(T)$ be the random number of steps, starting from $T$ until the MC hits $E$. Let also $N(H)$ be defined similarly. Finally, let $N'(T)$ be the number of steps after the second visit to $T$ until the MC hits $E$. Then,

$$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$$

where $Z = \{\text{first flip in } T \text{ is } H\}$. Since $Z$ and $N(H)$ are independent, and $Z$ and $N'(T)$ are independent, taking expectations, we get

$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

i.e.,

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$q := 1 - p$

$S$: Start
$H$: Last flip $= H$
$T$: Last flip $= T$
$E$: Done
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First Passage Time - Example 2

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You keep rolling a fair six-sided die until the sum of the last two rolls is 8.

Question: How many times do you have to roll the die before you stop, on average?

Spoiler Alert: Solution on next slide (but don't look: try to do it yourself first!)
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Example 3: Practice Exercise Solution

\[ \beta(S) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j); \]

\[ \beta(1) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j); \]

\[ \beta(i) = 1 + \frac{1}{6} \sum_{j=1, \ldots, 6}^{i} \beta(j), \quad i = 2, \ldots, 6. \]

Symmetry: \[ \beta(2) = \cdots = \beta(6) =: \gamma. \]

Also, \[ \beta(1) = \beta(S). \]

Thus, \[ \beta(S) = 1 + \left( \frac{5}{6} \right) \gamma + \beta(S) \frac{1}{6}; \]

\[ \gamma = 1 + \left( \frac{4}{6} \right) \gamma + \left( \frac{1}{6} \right) \beta(S). \]

\[ \Rightarrow \cdots \beta(S) = 8.4. \]
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\[ i = 2, \ldots, 6. \]

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⇒ \[ \beta(S) = 8/4. \]

\[ S = \text{Start}; \ E = \text{Done} \]
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\[ P(S, j) = 1/6, j = 1, \ldots, 6 \]
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The arrows out of 3, \ldots, 6 (not shown) are similar to those out of 2.
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First Step Equations

Let $X^n$ be a MC on $X$ and $A \subset X$.

Define $\text{T}_A = \min\{n \geq 0 | X^n \in A\}$.

Let $\beta(i) = E[\text{T}_A | X^0 = i]$, $i \in X$.

The FSE are $\beta(i) = 0$, $i \in A$ 

$\beta(i) = 1 + \sum_j P(i,j) \beta(j)$, $i / \in A$.
First Step Equations

Let \( X_n \) be a MC on \( \mathcal{X} \) and \( A \subset \mathcal{X} \).
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Summary

Markov Chains

Markov Chain:

\[ Pr[X_{n+1} = j | X_0, \ldots, X_n = i] = P(i, j) \]

First Passage Time:

\[ A \subset X; \beta(i) = E[T_A | X_0 = i]; \quad \beta(i) = 1 + \sum_j P(i, j) \beta(j); \]

FSE:

\[ \beta(i) = 1 + \sum_j P(i, j) \beta(j) \]

\[ \pi_n = \pi_0 P^n \]

\[ \pi \text{ is invariant iff } \pi P = \pi \]

Irreducible \[ \Rightarrow \] one and only one invariant distribution \[ \pi \]
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Probability part of the course: key takeaways?

What should I take away about probability from this course?

- Given the uncertainty around us, we should understand some probability. "Being precise about being imprecise."

- 4 key concepts:
  1. Learn from observations to revise our biases, given by the role of the prior; Bayes' Theorem.
  2. Confidence Intervals: CLT, Chebyshev Bounds, WLLN.
  4. Markov Chains: Sequence of RVs, $P[X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots] = P[X_{n+1} = x_{n+1} | X_n = x_n]$, Balance Equations.

- Quantifying our degree of certainty. This clear thinking invites us to question vague statements, and to convert them into precise ideas.
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Probability part of the course: key takeaways?

What should I take away about probability from this course?
I mean, after the final?

► Given the uncertainty around us, we should understand some probability. “Being precise about being imprecise.”

► 4 key concepts:

1. Learn from observations to revise our biases, given by the role of the prior; Bayes’ Theorem;
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Random Thoughts

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“There are three kinds of lies: lies, damned lies, and statistics.”
The numbers are applications and admissions of males and females to the two colleges of a university. Overall, the admission rate of male students is 80\% whereas it is only 51\% for female students. A closer look shows that the admission rate is larger for female students in both colleges. Female students happen to apply more to the college that admits fewer students.
Confusing Statistics: Simpson’s Paradox

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Statistics are often confusing:

▶ The average household annual income in the US is $72k. Yes, but the median is $52k.

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▶ The Texas sharpshooter fallacy. Look at people living close to power lines. You find clusters of cancers. You will also find such clusters when looking at people eating kale.

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Confirmation Bias

Confirmation bias is the tendency to search for, interpret, and recall information in a way that confirms one's beliefs or hypotheses, while giving disproportionately less consideration to alternative possibilities. Confirmation biases contribute to overconfidence in personal beliefs and can maintain or strengthen beliefs in the face of contrary evidence.

Three aspects:
▶ Biased search for information. E.g., ignoring articles that dispute your beliefs.
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There are two bags. One with 60% red balls and 40% blue balls; the other with the opposite fractions. One selects one of the two bags. As one draws balls one at a time, one asks people to declare whether they think one draws from the first or second bag. Surprisingly, people tend to be reinforced in their original belief, even when the evidence accumulates against it.
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▶ A judge rolls a die in the morning. In the afternoon, he has to sentence a criminal. Statistically, the sentence tends to be heavier if the outcome of the morning roll was high.

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Parting Thoughts

You have worked hard and learned a lot in this course! Proofs, Graphs, Stable Marriage, Mod(p), RSA, Reed-Solomon, Decidability, Probability, ... , HW option or Test-only option? How to handle stress, how to sleep less, how to keep smiling, ... , Difficult course? Yes! Useful? You bet! Finally, THANK YOU on behalf of Prof. Rao and me for persevering through this course! It has been an absolute pleasure! Let us also not forget to thank the dedicated EECS70 Staff: ▶ The Thrilling TAs ▶ The Terrific Tutors ▶ The Rigorous Readers ▶ The Amazing Assistants GOOD LUCK IN YOUR FINAL EXAM!!
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