Announcements

Will provide data on past performance for test-only versus homework on piazza and in class before you have to make final decision. In the meantime, at least consider doing homework 2. Time after class. I generally keep that time available for students, so catch me.

Questions?

Stable Marriage Problem

- Small town with \( n \) boys and \( n \) girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples...
- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.
Angelina prefers Brad to Billy-Bob.
Uh...oh.

So..

Produce a pairing where there is no running off!

**Definition:** A pairing is disjoint set of \( n \) boy-girl pairs.

Example: A pairing \( S = \{ (Brad, Jen); (BillyBob, Angelina) \} \).

**Definition:** A rogue couple \( b, g^* \) for a pairing \( S \):
\( b \) and \( g^* \) prefer each other to their partners in \( S \)

Example: Brad and Angelina are a rogue couple in \( S \).

A stable pairing??

Given a set of preferences.
Is there a stable pairing?
How does one find it?

Consider a single gender version: stable roommates.

\[
\begin{array}{cccc}
A & B & C & D \\
B & C & A & D \\
C & A & B & D \\
D & A & B & C \\
\end{array}
\]

C

D

A

B
The Traditional Marriage Algorithm.

Each Day:
1. Each boy proposes to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>

Day 1: A, B
Day 2: C
Day 3: B
Day 4: C
Day 5: B

Termination.

Every non-terminated day a boy crossed an item off the list.

Total size of lists? \( n \) boys, \( n \) length list. \( n^2 \)

Terminates in at most \( n^2 + 1 \) steps!

It gets better every day for girls..

Improvement Lemma: It just gets better for girls.

If on day \( t \) a girl \( g \) has a boy \( b \) on a string,
any boy, \( b' \), on \( g \)'s string for any day \( t' > t \)
is at least as good as \( b \).

Proof:
- \( P(k) \sim \) “boy on \( g \)'s string is at least as good as \( b \) on day \( t + k \)”
- \( P(0) \sim \) true. Girl has \( b \) on string.
- Assume \( P(k) \). Let \( b' \) be boy on string on day \( t + k \).

On day \( t + k + 1 \), boy \( b' \) comes back.

Girl can choose \( b' \), or do better with another boy, \( b'' \)
That is, \( b \leq b' \) by induction hypothesis. And \( b'' \) is better than \( b' \) by algorithm.

\( \Rightarrow \) Girl does at least as well as with \( b \).
- \( P(k) \sim P(k + 1) \). And by principle of induction.

Pairing when done.

Lemma: Every boy is matched at end.

Proof:
If not, a boy \( b \) must have been rejected \( n \) times.
Every girl has been proposed to by \( b \),
and Improvement lemma
\( \Rightarrow \) each girl has one boy on a string.
and each boy is on at most one string.

\( n \) girls and \( n \) boys. Same number of each.

\( \Rightarrow b \) must be on some girl's string!

Contradiction.

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by the traditional marriage algorithm.

Proof:
Assume there is a rogue couple; \((b, g^*)\)

\( \Rightarrow \) \( b \) likes \( g^* \) more than \( g \).

\( \Rightarrow \) \( b' \) likes \( g \) more than \( b \).

Boy \( b \) proposes to \( g^* \) before proposing to \( g \).
So \( g^* \) rejected \( b \) (since he moved on)
By Improvement lemma, \( g^* \) likes \( b' \) better than \( b \).

Contradiction!
<table>
<thead>
<tr>
<th>Good for boys? girls?</th>
<th>TMA is optimal!</th>
<th>How about for girls?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is the TMA better for boys? for girls?</td>
<td><strong>Theorem:</strong> TMA produces a boy-optimal pairing.</td>
<td><strong>Theorem:</strong> TMA produces girl-pessimal pairing.</td>
</tr>
<tr>
<td><strong>Definition:</strong> A pairing is ( x )-optimal if ( x )'s partner is its best partner in any stable pairing.</td>
<td><strong>Proof:</strong> Assume not: There is stable pairing where some boy does better.</td>
<td>( T ) – pairing produced by TMA.</td>
</tr>
<tr>
<td><strong>Definition:</strong> A pairing is ( x )-pessimal if ( x )'s partner is its worst partner in any stable pairing.</td>
<td><strong>Let</strong> I be first day a boy ( b ) gets rejected by his the optimal girl ( g ) who he is paired with in stable pairing ( S ).</td>
<td>( S ) – worse stable pairing for girl ( g ).</td>
</tr>
<tr>
<td><strong>Definition:</strong> A pairing is boy optimal if it is ( x )-optimal for all boys ( x ).</td>
<td><strong>TMA:</strong> ( b' ) - knocks ( b ) off of ( g )'s string on day ( t ) ( \implies g ) prefers ( b' ) to ( b )</td>
<td>In ( T ), ((g,b)) is pair.</td>
</tr>
<tr>
<td>...and so on for boy pessimal, girl optimal, girl pessimal.</td>
<td><strong>By choice of ( t ), ( b' ) prefers ( g ) to his partner in ( S ).</strong></td>
<td>In ( S ), ((g,b')) is pair.</td>
</tr>
<tr>
<td><strong>Check:</strong> The optimal partner for a boy must be first in his preference list.</td>
<td>( \implies b' ) prefers ( g ) to his partner ( g^* ) in ( S ).</td>
<td>( g ) likes ( b' ) less than she likes ( b ).</td>
</tr>
<tr>
<td>True? False? False!</td>
<td>Rogue couple for ( S ). So ( S ) is not a stable pairing. Contradiction.</td>
<td>( T ) is boy optimal, so ( b ) likes ( g ) more than his partner in ( S ).</td>
</tr>
<tr>
<td>Subtlety here: Best partner in any stable pairing. As well as you can do in a globally stable solution!</td>
<td>Notes: Not really induction. Structural statement: Boy optimality ( \implies ) Girl pessimal.</td>
<td></td>
</tr>
<tr>
<td><strong>Question:</strong> Is there a boy or girl optimal pairing? Is it possible: ( b )-optimal pairing different from the ( b' )-optimal pairing? Yes? No?</td>
<td>Used Well-Ordering principle...Induction.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quick Questions.</th>
<th>Residency Matching..</th>
<th>Don't go!</th>
</tr>
</thead>
<tbody>
<tr>
<td>How does one make it better for girls?</td>
<td>The method was used to match residents to hospitals.</td>
<td>Summary.</td>
</tr>
<tr>
<td>SMA - stable marriage algorithm. One side proposes.</td>
<td>Hospital optimal... ( \ldots ) until 1990's...Resident optimal.</td>
<td><strong>Summary:</strong></td>
</tr>
<tr>
<td>TMA - boys propose. Girls could propose. ( \implies ) ( \text{optimal for girls.} )</td>
<td>Another variation: couples.</td>
<td><strong>Don't go!</strong></td>
</tr>
</tbody>
</table>