

Announcements

Will provide data on past performance for test-only versus homework on piazza and in class before you have to make final decision.

In the meantime, at least consider doing homework 2.

Time after class. I generally keep that time available for students, so catch me.

Questions?

Stable Marriage Problem

- ▶ Small town with n boys and n girls.
- ▶ Each girl has a ranked preference list of boys.
- ▶ Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- ▶ Maximize total satisfaction.
- ▶ Maximize number of first choices.
- ▶ Maximize worse off.
- ▶ Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- ▶ Jennifer and Brad
- ▶ Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of n boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

Definition: A **rogue couple** b, g^* for a pairing S :
 b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S .

A stable pairing??

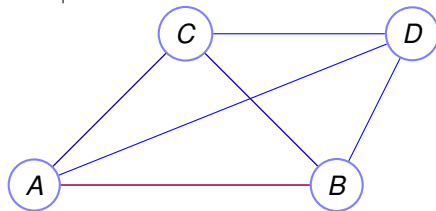
Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite girl on his list.
2. Each girl rejects all but her favorite proposer (whom she puts on a **string**.)
3. Rejected boy **crosses** rejecting girl off his list.

Stop when each girl gets exactly one proposal.

Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

	Boys				Girls		
A	X	2	3	1	C	A	B
B	X	X	3	2	A	B	C
C	X	1	3	3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B	A	X , C	C	C
2	C	B, C	B	A, B	A
3					B

Termination.

Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

Terminates in at most $n^2 + 1$ steps!

It gets better every day for girls..

Improvement Lemma: It just gets better for girls.

If on day t a girl g has a boy b on a string,
any boy, b' , on g 's string for any day $t' > t$
is at least as good as b .

Proof:

$P(k)$ - "boy on g 's string is at least as good as b on day $t + k$ "

$P(0)$ - true. Girl has b on string.

Assume $P(k)$. Let b' be boy **on string** on day $t + k$.

On day $t + k + 1$, boy b' comes back.

Girl can choose b' , or do better with another boy, b''

That is, $b \leq b'$ by induction hypothesis.

And b'' is better than b' **by algorithm**.

\implies Girl does at least as well as with b .

$P(k) \implies P(k + 1)$. And by principle of induction. □

Pairing when done.

Lemma: Every boy is matched at end.

Proof:

If not, a boy b must have been rejected n times.

Every girl has been proposed to by b ,
and **Improvement lemma**

\implies each girl has **one** boy on a string.

and each boy is on at most **one** string.

n girls and n boys. Same number of each.

\implies b must be on some girl's string!

Contradiction.



Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by the traditional marriage algorithm.

Proof:

Assume there is a rogue couple; (b, g^*)

$b^* \text{ ————— } g^*$

b likes g^* more than g .

$b \text{ ————— } g$

g^* likes b more than b^* .

Boy b proposes to g^* before proposing to g .

So g^* rejected b (since he moved on)

By improvement lemma, g^* likes b^* better than b .

Contradiction!



Good for boys? girls?

Is the TMA better for boys? for girls?

Definition: A **pairing is x -optimal** if x 's partner is its best partner in any **stable** pairing.

Definition: A **pairing is x -pessimal** if x 's partner is its worst partner in any **stable** pairing.

Definition: A **pairing is boy optimal** if it is x -optimal for **all** boys x .

..and so on for boy pessimal, girl optimal, girl pessimal.

Check:

The optimal partner for a boy must be first in his preference list.

True? False? False!

Subtlety here: Best partner in any **stable** pairing.

As well as you can do in a globally stable solution!

Question: Is there a boy or girl optimal pairing?

Is it possible:

b -optimal pairing different from the b' -optimal pairing!

Yes? No?

TMA is optimal!

For boys? For girls?

Theorem: TMA produces a boy-optimal pairing.

Proof:

Assume not:

There is stable pairing where some boy does better.

Let t be first day a boy b gets rejected
by his the optimal girl g who he is paired with
in stable pairing S .

TMA: b^* - knocks b off of g 's string on day $t \implies g$ prefers b^* to b

By choice of t , b^* prefers g to his partner in S .

$\implies b^*$ prefers g to his partner g^* in S .

Rogue couple for S .

So S is not a stable pairing. Contradiction. □

Notes: S - stable. $(b^*, g^*) \in S$. But (b^*, g) is rogue couple!

Used Well-Ordering principle...Induction.

How about for girls?

Theorem: TMA produces girl-pessimal pairing.

T – pairing produced by TMA.

S – worse **stable pairing** for girl g .

In T , (g, b) is pair.

In S , (g, b^*) is pair.

g likes b^* less than she likes b .

T is boy optimal, so b likes g more than his partner in S .

(g, b) is Rogue couple for S

S is not stable.

Contradiction.



Notes: Not really induction.

Structural statement: Boy optimality \implies Girl pessimality.

Quick Questions.

How does one make it better for girls?

SMA - stable marriage algorithm. One side proposes.

TMA - boys propose.

Girls could propose. \implies optimal for girls.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Another variation: couples.

Don't go!

Summary.

▶ [Link](#)