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Questions?

Stable Marriage Problem

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- Small town with n boys and n girls.

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- Each girl has a ranked preference list of boys.

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Stable Marriage Problem

- Small town with n boys and n girls.
- Each girl has a ranked preference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.

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- Maximize total satisfaction.
- Maximize number of first choices.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

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- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

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Brad prefers Angelina to Jennifer.

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The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

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Definition: A **pairing** is disjoint set of n boy-girl pairs.

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Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

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Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

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Produce a pairing where there is no running off!

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Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S.

A stable pairing??

Given a set of preferences.

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Is there a stable pairing?
How does one find it?

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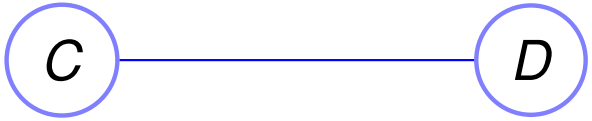
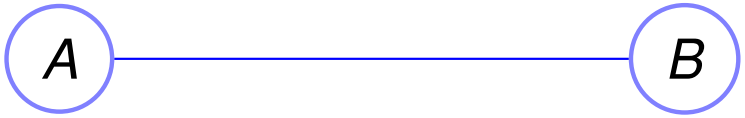
Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C





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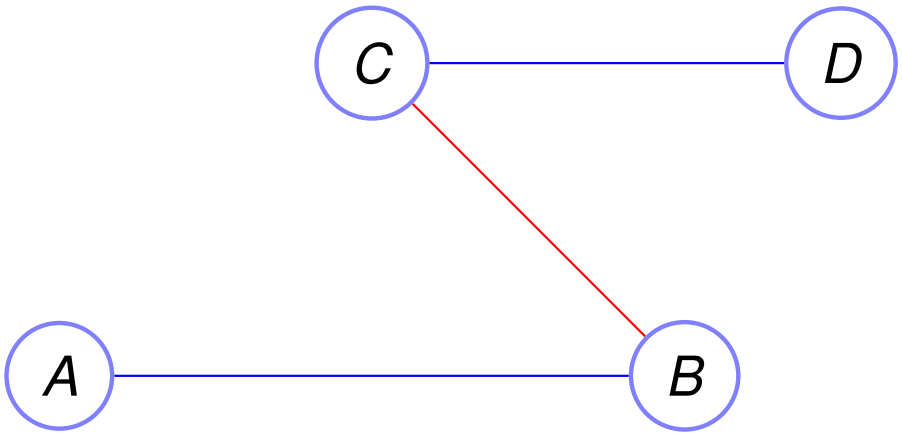

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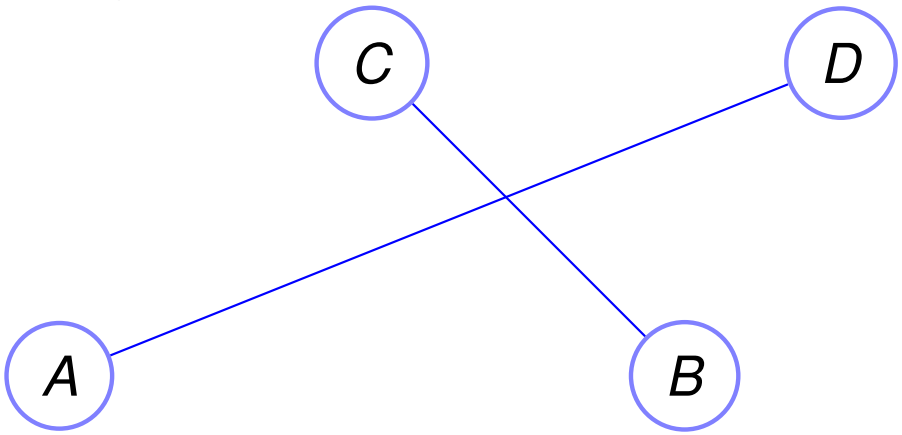

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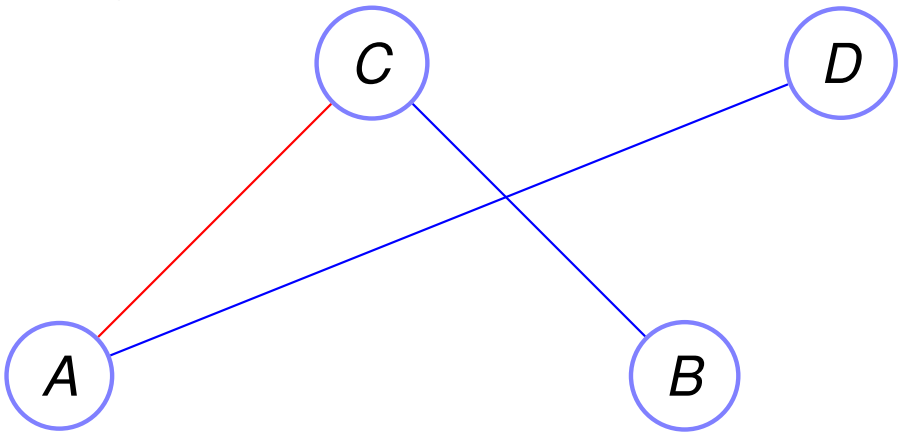

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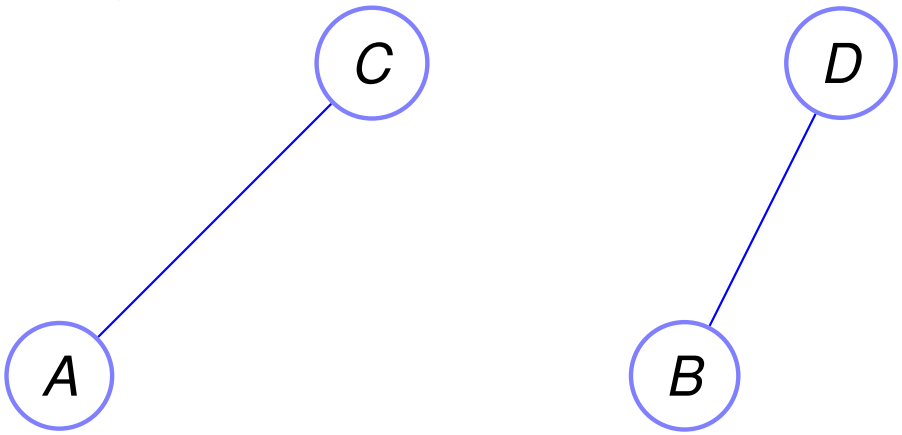

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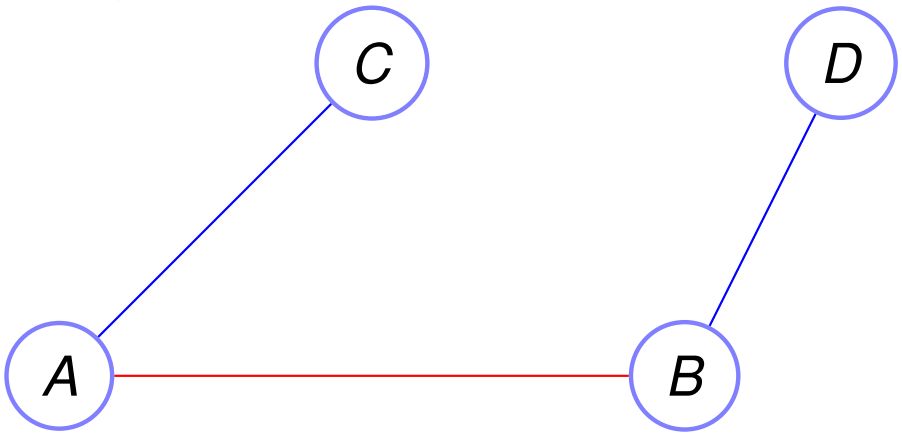

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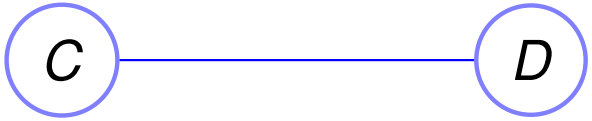

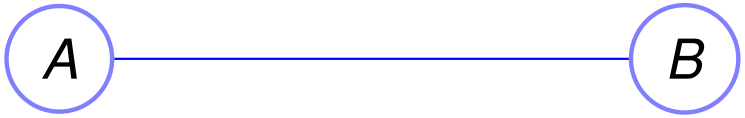

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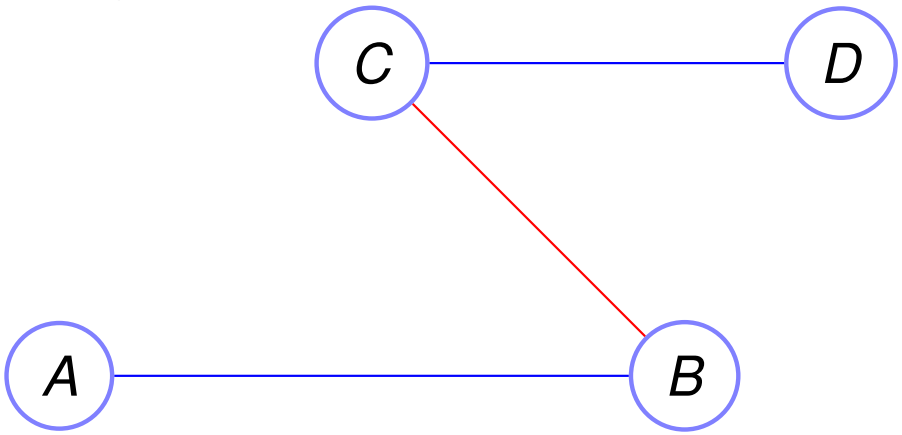

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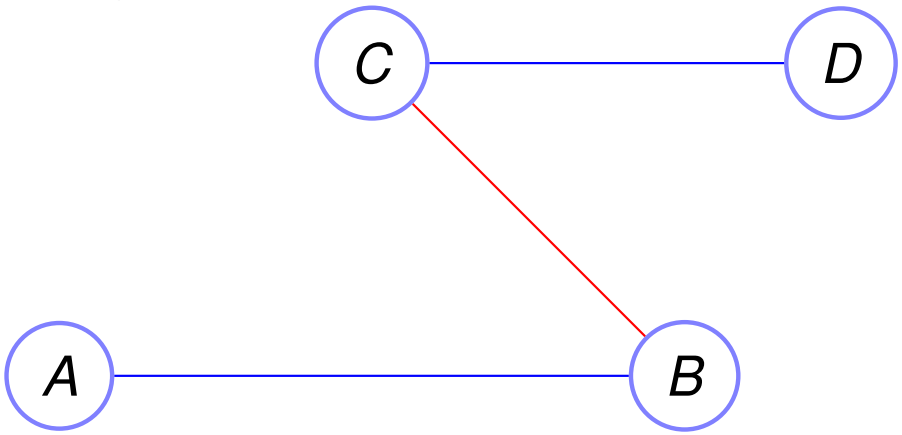

The Traditional Marriage Algorithm.

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Each Day:

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1. Each boy **proposes** to his favorite girl on his list.

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3. Rejected boy crosses rejecting girl off his list.

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Stop when each girl gets exactly one proposal.

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Stop when each girl gets exactly one proposal.
Does this terminate?

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...produce a pairing?

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Do boys or girls do “better”?

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Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

	Boys		
A	1	2	3
B	1	2	3
C	2	1	3

	Girls		
1	C	A	B
2	A	B	C
3	A	C	B

Example.

	Boys		
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	Girls		
1	C	A	B
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3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

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A	1	2	3
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C	2	1	3

	Girls		
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

	Boys		
A	1	2	3
B	~~1~~	2	3
C	2	1	3

	Girls		
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
3					

Example.

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	Girls		
1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, ~~C~~			
3					

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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	A , C		
2	C	B, ~~C~~	B		
3					

Example.

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3					

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2	C	B, C	B	A, B	A
3					B

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	Day 1	Day 2	Day 3	Day 4	Day 5
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3					B

Termination.

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Every non-terminated day a boy **crossed** an item off the list.

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Total size of lists?

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Total size of lists? n boys, n length list.

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Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

Termination.

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Total size of lists? n boys, n length list. n^2

Terminates in at most $n^2 + 1$ steps!

It gets better every day for girls..

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Improvement Lemma: It just gets better for girls.

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If on day *t* a girl *g* has a boy *b* on a string,

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If on day t a girl g has a boy b on a string,
any boy, b', on g's string for any day $t' > t$

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If on day t a girl g has a boy b on a string,
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Proof:
$P(k)$- - "boy on g's string is at least as good as b on day $t + k$"

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$P(0)$– true. Girl has b on string.

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Girl can choose b',

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Girl can choose b', or do better with another boy, b''

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That is, $b \leq b'$ by induction hypothesis.

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$\Longrightarrow$ Girl does at least as well as with b.

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Pairing when done.

Lemma: Every boy is matched at end.

Pairing when done.

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Proof:

Pairing when done.

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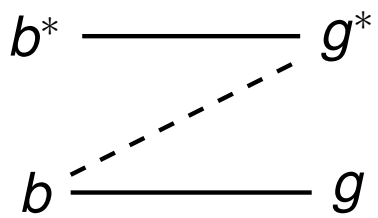

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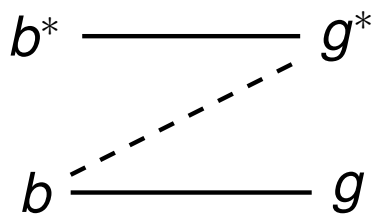

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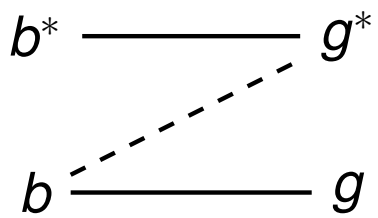

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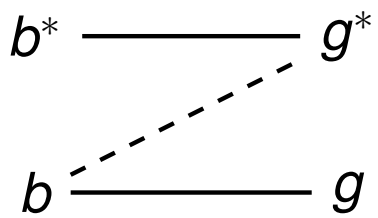

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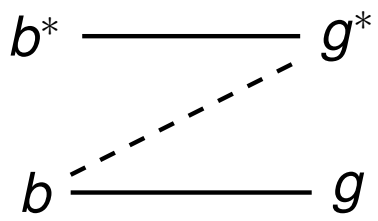

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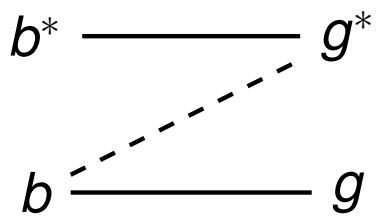

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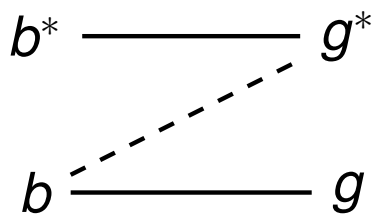

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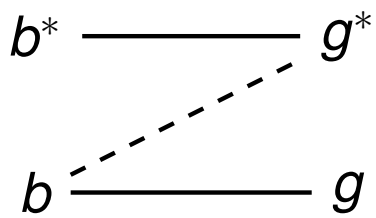

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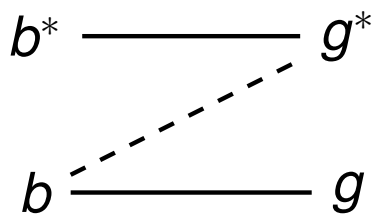

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Theorem: TMA produces a boy-optimal pairing.

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For boys? For girls?

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Assume not:
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▸ Link