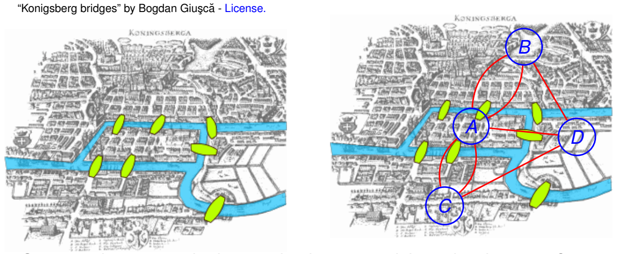
Lecture 5: Graphs.

Graphs!
  Euler
  Definitions: model.
  Fact!
  Euler Again!!
Planar graphs.
  Euler Again!!!!

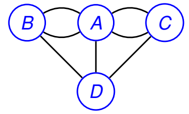
Konigsberg bridges problem.

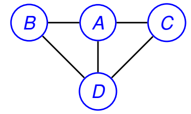
Can you make a tour visiting each bridge exactly once?



Can you draw a tour in the graph where you visit each edge once?
Yes? No?
We will see!

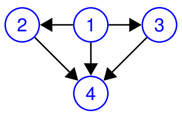
Graphs: formally.





Graph: $G = (V, E)$.
  V - set of vertices.
    $\{A, B, C, D\}$
  $E \subseteq V \times V$ - set of edges.
    $\{\{A, B\}, \{A, B\}, \{A, C\}, \{B, C\}, \{B, D\}, \{B, D\}, \{C, D\}\}$.
For CS 70, usually simple graphs.
  No parallel edges.
Multigraph above.

Directed Graphs



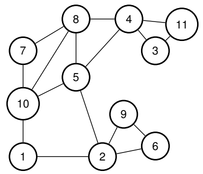
$G = (V, E)$.
  V - set of vertices.
    $\{1, 2, 3, 4\}$
  E ordered pairs of vertices.
    $\{(1,2), (1,3), (1,4), (2,4), (3,4)\}$

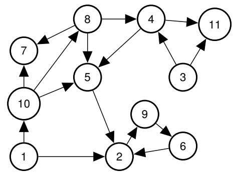
One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed? Undirected?
  Friends. Undirected.
  Likes. Directed.

Graph Concepts and Definitions.

Graph: $G = (V, E)$
  neighbors, adjacent, degree, incident, in-degree, out-degree





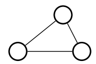
Neighbors of 10? 1,5,7, 8.
  u is neighbor of v if $(u, v) \in E$.
Edge $(10, 5)$ is incident to vertex 10 and vertex 5.
  Edge (u, v) is incident to u and v.
Degree of vertex 1? 2
  Degree of vertex u is number of incident edges.
  Equals number of neighbors in simple graph.
Directed graph: In-degree of 10? 1   Out-degree of 10? 3

Quick Proof.

The sum of the vertex degrees is equal to

(A) the total number of vertices, $|V|$.
(B) the total number of edges, $|E|$.
(C) What?

Not (A)! Triangle.



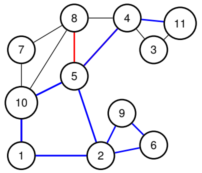
Not (B)! Triangle.

What? For triangle number of edges is 3, the sum of degrees is 6.

Could it always be...$2|E|$? ..or $2|V|$?

  How many incidences does each edge contribute? 2.
    $2|E|$ incidences are contributed in total!
  What is degree v? incidences contributed to v!
   sum of degrees is total incidences ... or $2|E|$.
Thm: Sum of vertex degress is $2|E|$.

Paths, walks, cycles, tour.



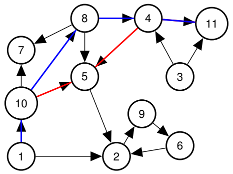
A path in a graph is a sequence of edges.

Path? $\{1,10\}$, $\{8,5\}$, $\{4,5\}$? No!
Path? $\{1,10\}$, $\{10,5\}$, $\{5,4\}$, $\{4,11\}$? Yes!

Path: $(v_1,v_2),(v_2,v_3),\ldots(v_{k-1},v_k)$.
Quick Check! Length of path? k vertices or $k-1$ edges.
Cycle: Path with $v_1 = v_k$. Length of cycle? $k-1$ vertices and edges!
Path is usually simple. No repeated vertex!
Walk is sequence of edges with possible repeated vertex or edge.
Tour is walk that starts and ends at the same node.

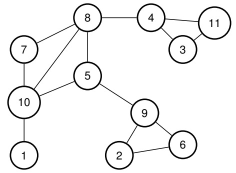
Quick Check!
Path is to Walk as Cycle is to ?? Tour!

Directed Paths.



Path: $(v_1,v_2),(v_2,v_3),\ldots(v_{k-1},v_k)$.
Paths, walks, cycles, tours ... are analagous to undirected now.

Connectivity



u and v are connected if there is a path between u and v.

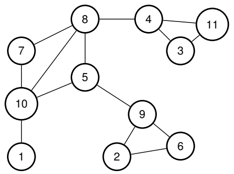
A connected graph is a graph where all pairs of vertices are connected.

If one vertex x is connected to every other vertex.
Is graph connected? Yes? No?

Proof: Use path from u to x and then from x to v. $\square$

May not be simple!
Either modify definition to walk.
Or cut out cycles. .

Understanding Definition.



Is graph above connected? Yes!

How about now? No!

Connected Components? $\{1\},\{10,7,5,8,4,3,11\},\{2,9,6\}$.

Connected component - maximal set of connected vertices.

Quick Check: Is $\{10,7,5\}$ a connected component? No.

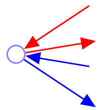
Not maximal.

Finally..back to Euler!

An Eulerian Tour is a tour that visits each edge exactly once.

Theorem: Any undirected graph has an Eulerian tour if and only if all vertices have even degree and is connected.

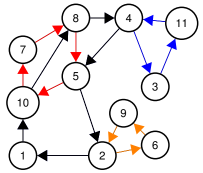
Proof of only if: Eulerian $\implies$ connected and all even degree.
Eulerian Tour is connected so graph is connected.
Tour enters and leaves vertex v on each visit.
Uses two incident edges per visit. Tour uses all incident edges.
Therefore v has even degree. $\square$



When you enter, you leave.
For starting node, tour leaves first ....then enters at end.

Finding a tour!

Proof of if: Even + connected $\implies$ Eulerian Tour.
We will give an algorithm. First by picture.



1. Take a walk starting from v (1) on "unused" edges
... till you get back to v.
2. Remove tour, C.
3. Let $G_1,\ldots,G_k$ be connected components.
   Each is touched by C.
   Why? G was connected.
   Let v_i be (first) node in G_i touched by C.
   Example: $v_1 = 1$, $v_2 = 10$, $v_3 = 4$, $v_4 = 2$.
4. Recurse on $G_1,\ldots,G_k$ starting from v_i
5. Splice together.
   1,10,7,8,5,10 ,8,4,3,11,4 5,2,6,9,2 and to 1!

Recursive/Inductive Algorithm.

1. Take a walk from arbitrary node v, until you get back to v.

Claim: Do get back to v!
Proof of Claim: Even degree. If enter, can leave except for v. $\square$

2. Remove cycle, C, from G.

Resulting graph may be disconnected. (Removed edges!)
Let components be $G_1, \ldots, G_k$.
Let v_i be first vertex of C that is in G_i.
Why is there a v_i in C?
G was connected $\implies$
a vertex in G_i must be incident to a removed edge in C.

Claim: Each vertex in each G_i has even degree and is connected.
Prf: Tour C has even incidences to any vertex v. $\square$

3. Find tour T_i of G_i starting/ending at v_i. Induction.
4. Splice T_i into C where v_i first appears in C.

Visits every edge once:
Visits edges in C exactly once.
By induction for all edges in each G_i. $\square$

TA/Professor Ramchandran opinion

Do your homework! Darnit.
Make them do their homework! It is good for them.
Even if it makes little difference on test performance.
E.g. One test prep method: grind through old exams.
You still learn.
Tests not necessarily a measure of learning.
Indispensable learning tool: go toe-to-toe with problems!

HWs force learning depth and discipline....
like a gourmet meal vs. fast-food (cramming for test-only)....

Administration Time!

Well admin time!

Must choose homework option or test only: soon after recieving hw 1 scores.

Test Option: don't have to do homework. Yes!!
Should do homework. No need to write up.
Homework Option: have to do homework. Bummer!

The truth: mostly test, both options!
Variance mostly in exams for A/B range.
most homework students get near perfect scores on homework.

How will I do?

Mostly up to you.

A few details.

Welcome to turn in homework in test-only option.

Changes:
80% max on homework.
Test Only Reflection.

Professor Rao. What should I do?

Whatever you think best for you.