Today.

Types of graphs.
Types of graphs.

Complete Graphs.
Trees.
Planar Graphs.
Today.

Types of graphs.

Complete Graphs.
Trees.
Planar Graphs.
Complete Graph.

\( K_n \) complete graph on \( n \) vertices.
$K_n$ complete graph on $n$ vertices.
All edges are present.
Complete Graph.

\( K_n \) complete graph on \( n \) vertices.
All edges are present.
Everyone is my neighbor.
Complete Graph.

$K_n$ complete graph on $n$ vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.
Complete Graph.

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How many edges?
Complete Graph.

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Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n - 1$ edges.
Complete Graph.

$K_n$ complete graph on $n$ vertices.
All edges are present.
Everyone is my neighbor.
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How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1)$.
Complete Graph.

\( K_n \) complete graph on \( n \) vertices.
All edges are present.
Everyone is my neighbor.
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How many edges?
Each vertex is incident to \( n - 1 \) edges.
Sum of degrees is \( n(n - 1) \).
\[ \Rightarrow \text{Number of edges is } \frac{n(n - 1)}{2}. \]
**Complete Graph.**

$K_n$ complete graph on $n$ vertices.
- All edges are present.
- Everyone is my neighbor.
- Each vertex is adjacent to every other vertex.

How many edges?
- Each vertex is incident to $n-1$ edges.
- Sum of degrees is $n(n-1)$.
  \[ \sum \text{deg} = n(n-1) \]
  \[ \implies \text{Number of edges is } n(n-1)/2. \]
- Remember sum of degree is $2|E|$. 

$K_5$ is not planar.
$K_5$ is not planar.
Cannot be drawn in the plane without an edge crossing!
$K_5$ is not planar.
Cannot be drawn in the plane without an edge crossing!
Prove it!
$K_5$ is not planar.  
Cannot be drawn in the plane without an edge crossing! 
Prove it! We will!
A Tree, a tree.

Graph $G = (V, E)$.
Binary Tree!

More generally.
Trees.

Definitions:

A connected graph without a cycle.

A connected graph with $|V| - 1$ edges.

A connected graph where any edge removal disconnects it.

A connected graph where any edge addition creates a cycle.

Some trees with no cycle and connected?

Yes.

$|V| - 1$ edges and connected?

Yes.

Removing any edge disconnects it.

Harder to check. But yes.

Adding any edge creates a cycle.

Harder to check. But yes.

To tree or not to tree!
Trees.

Definitions:

A connected graph without a cycle.
Trees.

Definitions:
- A connected graph without a cycle.
- A connected graph with $|V| - 1$ edges.
Trees.

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Some trees. Are they connected and have no cycle? Yes. Are they connected and have $|V| - 1$ edges? Yes. Is it harder to check if removing any edge disconnects it? Yes. Is it harder to check if adding any edge creates a cycle? Yes. To tree or not to tree!
Trees.

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- A connected graph without a cycle.
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Some trees.

no cycle and connected?
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Some trees.

<table>
<thead>
<tr>
<th>Cycle</th>
<th>No cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Trees.

Definitions:

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- A connected graph with \(|V| − 1\) edges.
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Some trees.

![Diagram of trees](image)

- no cycle and connected? Yes.
- \(|V| − 1\) edges and connected? Yes.
- removing any edge disconnects it. Harder to check. but yes.
- Adding any edge creates cycle.
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- A connected graph where any edge addition creates a cycle.

Some trees.

- [Diagram of a tree with no cycle and connected.]
- [Diagram of a tree with $|V| - 1$ edges and connected.]
- [Diagram of a tree where removing any edge disconnects it.]
- [Diagram of a tree where adding any edge creates a cycle.]

- no cycle and connected? Yes.
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To tree or not to tree!
Theorem:
“G connected and has $|V| - 1$ edges” $\equiv$
“G is connected and has no cycles.”
Equivalence of Definitions.

**Theorem:**
“G connected and has $|V| - 1$ edges” $\equiv$
“G is connected and has no cycles.”

**Lemma:** If $v$ is degree 1 in connected $G$, then $G - v$ is connected.

**Proof:**
For $x \neq v, y \neq v \in V$,
Theorem:
“\(G\) connected and has \(|V| - 1\) edges” \(\equiv\)
“\(G\) is connected and has no cycles.”

Lemma: If \(v\) is degree 1 in connected \(G\), then \(G - v\) is connected.

Proof:
For \(x \neq v, y \neq v \in V\),
there is path between \(x\) and \(y\) in \(G\) since connected.
Equivalence of Definitions.

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“G connected and has $|V| - 1$ edges” $\equiv$
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Equivalence of Definitions.

**Theorem:**
“$G$ connected and has $|V| - 1$ edges” $\equiv$
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$\implies$ $G - v$ is connected.
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**Theorem:**
“G connected and has $|V| - 1$ edges” ≡
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Proof of only if.

**Thm:**
“$G$ connected and has $|V| - 1$ edges” ≡
“$G$ is connected and has no cycles.”

**Proof of $\Rightarrow$:**

By induction on $|V|$.

**Base Case:**
$|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

**Induction Step:**
Claim: There is a degree 1 node.

Proof: First, connected $\Rightarrow$ every vertex degree $\geq 1$.

Sum of degrees is $2|V| - 2$.

Average degree $2 - 2/|V|$.

Not everyone is bigger than average!

By degree 1 removal lemma, $G - v$ is connected.

$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction $\Rightarrow$ no cycle in $G - v$.

And no cycle in $G$ since degree 1 cannot participate in cycle.
Proof of only if.

Thm:
"G connected and has \(|V| − 1\) edges" \(\equiv\)
"G is connected and has no cycles."

Proof of \(\implies\) : By induction on \(|V|\).
Proof of only if.

**Thm:**
“G connected and has $|V| - 1$ edges” $\equiv$
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**Proof of $\implies$:** By induction on $|V|$.
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Proof of only if.

Thm:
"G connected and has $|V| - 1$ edges" \equiv
"G is connected and has no cycles."

Proof of $\implies$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\implies$ every vertex degree $\geq 1$.
Sum of degrees is $2|V| - 2$
Average degree $2 - 2/|V|$
Proof of only if.

Thm: “G connected and has $|V| - 1$ edges” $\equiv$ “G is connected and has no cycles.”

Proof of $\Rightarrow$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\Rightarrow$ every vertex degree $\geq 1$.
- Sum of degrees is $2|V| - 2$
- Average degree $2 - 2/|V|$
- Not everyone is bigger than average!
Proof of only if.

Thm:
“G connected and has $|V| - 1$ edges” $\equiv$
“G is connected and has no cycles.”

Proof of $\implies$ : By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

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Claim: There is a degree 1 node.
   Proof: First, connected $\implies$ every vertex degree $\geq 1$.
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“$G$ connected and has $|V| - 1$ edges” $\equiv$
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By induction on $|V|$.

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By degree 1 removal lemma, $G - v$ is connected.

$\square$
Proof of only if.

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$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction
Thm:
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Proof of $\implies$ : By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

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By degree 1 removal lemma, $G - v$ is connected.
$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction
$\implies$ no cycle in $G - v$. 
Proof of only if.

**Thm:**
“$G$ connected and has $|V| - 1$ edges” $\equiv$
“$G$ is connected and has no cycles.”

**Proof of $\equiv$:** By induction on $|V|$. 

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$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction $\implies$ no cycle in $G - v$.

And no cycle in $G$ since degree 1 cannot participate in cycle.
Proof of only if.

Thm: 
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And no cycle in $G$ since degree 1 cannot participate in cycle.
Thm:
“G is connected and has no cycles”
\[ \implies \text{“G connected and has } |V| - 1 \text{ edges”} \]

Proof:

Walk from a vertex using untraversed edges. Until get stuck.

Claim: Degree 1 vertex.

Proof of Claim:
Can't visit more than once since no cycle.
Entered.
Didn't leave.
Only one incident edge.
Removing node doesn't create cycle.
Removing degree 1 node doesn't disconnect from Degree 1 lemma.

By induction
\[ G - v \text{ has } |V| - 2 \text{ edges.} \]
\[ G \text{ has one more or } |V| - 1 \text{ edges.} \]
**Proof of if**

**Thm:**
“G is connected and has no cycles”
\[ \implies \text{“G connected and has } |V| - 1 \text{ edges”} \]

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Walk from a vertex using untraversed edges.
Proof of if

Thm:
“G is connected and has no cycles”

⇒ “G connected and has $|V| - 1$ edges”

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\[\Rightarrow \text{“G connected and has } |V| - 1 \text{ edges”}\]

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Can’t visit more than once since no cycle.
Entered. Didn’t leave. Only one incident edge.
Removing node doesn’t create cycle.
Proof of if

Thm:
“G is connected and has no cycles”
\[\implies\] “G connected and has \(|V| - 1\) edges”

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.

Claim: Degree 1 vertex.

Proof of Claim:
Can’t visit more than once since no cycle.
Entered. Didn’t leave. Only one incident edge.
Removing node doesn’t create cycle.
New graph is connected.
Proof of if

Thm:
“G is connected and has no cycles”
\[ \implies \text{“G connected and has } |V| - 1 \text{ edges”} \]

Proof:
Walk from a vertex using untraversed edges.
Until get stuck.

Claim: Degree 1 vertex.

Proof of Claim:
Can’t visit more than once since no cycle.
Entered. Didn’t leave. Only one incident edge.

Removing node doesn’t create cycle.
New graph is connected.
Removing degree 1 node doesn’t disconnect from Degree 1 lemma.
Proof of if

**Thm:**
“G is connected and has no cycles”
⇒ “G connected and has |V| – 1 edges”

**Proof:**
Walk from a vertex using untraversed edges.
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**Claim:** Degree 1 vertex.

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Can’t visit more than once since no cycle.
Entered. Didn’t leave. Only one incident edge.

Removing node doesn’t create cycle.
New graph is connected.
Removing degree 1 node doesn’t disconnect from Degree 1 lemma.
By induction \( G - v \) has \( |V| - 2 \) edges.
Thm:
“G is connected and has no cycles”
\[\implies \text{“G connected and has } |V| - 1 \text{ edges”}\]

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Walk from a vertex using untraversed edges. Until get stuck.

Claim: Degree 1 vertex.

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Can’t visit more than once since no cycle.
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Removing node doesn’t create cycle.
New graph is connected.
Removing degree 1 node doesn’t disconnect from Degree 1 lemma.
By induction \( G - v \) has \( |V| - 2 \) edges.
\( G \) has one more or \( |V| - 1 \) edges.
Thm:
“G is connected and has no cycles”
\[ \implies \text{ "G connected and has } |V| - 1 \text{ edges"} \]

Proof:
Walk from a vertex using untraversed edges.
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Can’t visit more than once since no cycle.
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Removing degree 1 node doesn’t disconnect from Degree 1 lemma.
By induction \( G - v \) has \( |V| - 2 \) edges.
\( G \) has one more or \( |V| - 1 \) edges.
Tree's fall apart.

**Thm:** There is one vertex whose removal disconnects $|V|/2$ nodes from each other.

Idea of proof.
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Idea of proof.
Point edge toward bigger side.
Tree’s fall apart.

**Thm:** There is one vertex whose removal disconnects $|V|/2$ nodes from each other.

Idea of proof.
Point edge toward bigger side.
Remove center node.
**Thm:** There is one vertex whose removal disconnects $|V|/2$ nodes from each other.

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Idea of proof.
Point edge toward bigger side.
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Planar graphs.

A graph that can be drawn in the plane without edge crossings.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar?

Yes for Triangle.

Four node complete?

Yes.

Five node complete or $K_5$?

No!

Why?

Later.

Two to three nodes, bipartite?

Yes.

Three to three nodes, complete/bipartite or $K_3,3$?

No.

Why?

Later.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete?


Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or K_3,3. No. Why? Later.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No!
Planar graphs.

A graph that can be drawn in the plane without edge crossings.

Planar? Yes for Triangle.
Four node complete? Yes.
Five node complete or $K_5$? No!
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Euler’s Formula.

Faces: connected regions of the plane.

How many faces for triangle?

2

Complete on four vertices or $K_4$?

4

Bipartite, complete two/three or $K_2,3$?

3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

Euler’s Formula: Connected planar graph has $v + f = e + 2$.

Triangle:

$3 + 2 = 3 + 2$

$K_4$:

$4 + 4 = 6 + 2$

$K_2,3$:

$5 + 3 = 6 + 2$

Examples = 3!

Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for triangle?
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for triangle? 2

Examples = 3!

Proven! Not!!!!
Euler’s Formula.

Faces: connected regions of the plane.

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Examples = 3!

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Euler’s Formula.

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Examples = 3!

Proven! Not!!
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Euler’s Formula.

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**Euler’s Formula:** Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2!$
$K_4$: $4 + 4 = 6 + 2!$
Euler’s Formula.

Faces: connected regions of the plane.

How many faces for
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Examples = 3!
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Examples = 3! Proven! Not!!!!
Euler and Polyhedron.

Greeks knew formula for polyhedron.
Euler and Polyhedron.

Greeks knew formula for polyhedron.
Euler and Polyhedron.

Greeks knew formula for polyhedron.

Faces?

\[
\begin{align*}
\text{Faces} & = 6 \\
\text{Edges} & = 12 \\
\text{Vertices} & = 8
\end{align*}
\]

Euler: Connected planar graph:

\[v + f = e + 2\]

Greeks couldn't prove it.

Induction?

Remove vertex for polyhedron?

Polyhedron without holes \(\equiv\) Planar graphs.

Surround by sphere.

Project from point inside polytope onto sphere.

Sphere \(\equiv\) Plane!

Topologically.

Euler proved formula thousands of years later!
Euler and Polyhedron.

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Faces? 6.   Edges?
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**Euler:** Connected planar graph: $v + f = e + 2$.

$8 + 6 = 12 + 2$. 
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**Euler:** Connected planar graph: \( v + f = e + 2. \)

\[ 8 + 6 = 12 + 2. \]

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Sphere $\equiv$ Plane! Topologically.

Euler proved formula thousands of years later!
Euler and planarity of $K_5$ and $K_{3,3}$

We consider graphs where $v \geq 3$. Each face is adjacent to edge at least 3 times for simple graph. Each edge is adjacent to (at most) two faces.

$3f \leq 2e$ for any planar graph with more than 2 vertices.

\[\Rightarrow e \leq \frac{3v}{2} - 3\]

$K_5$: Edges? 4 + 3 + 2 + 1 = 10.

Vertices? 5.

10 \n\not< \n\frac{3(5)}{2} - 6 = 9.

$\Rightarrow K_5$ is not planar.
Euler and planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.

$K_5$ edges: $4 + 3 + 2 + 1 = 10$.

$K_5$ vertices: 5.

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\[ 10 \not\leq 3(5) - 6 = 9 \]
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$\implies 3f \leq 2e$
Euler and planarity of $K_5$ and $K_{3,3}$

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Euler and planarity of $K_5$ and $K_{3,3}$

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Euler and planarity of $K_5$ and $K_{3,3}$

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... or $\frac{2}{3}e \geq f$. 

K5

K3,3
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... or $\frac{2}{3}e \geq f$.

$+\text{ Euler: } v + \frac{2}{3}e \geq e + 2$
Euler and planarity of $K_5$ and $K_{3,3}$

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$+ $ Euler: $v + \frac{2}{3} e \geq e + 2 \implies e \leq 3v - 6$
Euler and planarity of $K_5$ and $K_{3,3}$

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$K_5$
Euler and planarity of $K_5$ and $K_{3,3}$

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$K_5$ Edges?
Euler and planarity of $K_5$ and $K_{3,3}$

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Euler: $v + f = e + 2$ for connected planar graph.

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Euler and planarity of $K_5$ and $K_{3,3}$

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Each face is adjacent to edge at least 3 times for simple graph.
$\geq 3f$ face-edge adjacencies.
Each edge is adjacent to (at most) two faces.
$\leq 2e$ face-edge adjacencies.
$\implies 3f \leq 2e$ for any planar graph with more than 2 vertices
... or $\frac{2}{3}e \geq f$.

+ Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_5$ Edges? $4 + 3 + 2 + 1 = 10$. Vertices?
Euler and planarity of $K_5$ and $K_{3,3}$

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$10 \not\leq 3(5) - 6 = 9$. 
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$10 \not\leq 3(5) - 6 = 9. \implies K_5$ is not planar.
$K_{3,3}$ non-planarity.

\[
\begin{align*}
\text{Euler:} & \quad v + 2e \geq e + 2 = \Rightarrow e \leq 3v - 6 \\
\text{Vertices:} & \quad 9, 3 \\
\text{Edges:} & \quad 9, 3 \\
\text{Sure!} & \quad 9 \not\leq 2(6) - 4 \\
\text{Planar?} & \quad No. \\
\text{No cycles that are triangles.} & \\
\text{Cycles of length } \geq 4 & \\
\text{At least 4 } f \text{ face-edge adjacencies, and at most 2 } e. & \\
\text{... } 4 f \leq 2 e & \\
\text{for any bipartite planar graph.} & \\
\text{Euler:} & \quad v + 1/2 e \geq e + 2 = \Rightarrow e \leq 2v - 4 \text{ for bipartite planar graph} \\
\end{align*}
\]
$K_{3,3}$ non-planarity.

\[ v + \frac{2}{3} e \geq e + 2 \]

Planar? No.

No cycles that are triangles.

Cycles of length $\geq 4$.

At least 4 face-edge adjacencies, and at most 2 for any bipartite planar graph.

Euler: $v + \frac{2}{3} e \geq e + 2$
$K_{3,3}$ non-planarity.

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$
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$K_{3,3}$?
$K_{3,3}$ non-planarity.

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$K_{3,3}$? Edges?
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$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$?
$K_{3,3}$ non-planarity.

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Planar? No.
No cycles that are triangles.
\( K_{3,3} \) non-planarity.

\[
Euler: v + \frac{2}{3}e \geq e + 2 \quad \Longrightarrow \quad e \leq 3v - 6
\]

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  Cycles of length $\geq 4$.
At least $4f$ face-edge adjacencies,
\( K_{3,3} \) non-planarity.

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\( K_{3,3} \)? Edges? 9. Vertices. 6. \( 9 \leq 3(6) - 6 \)? Sure!

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No cycles that are triangles.
  Cycles of length \( \geq 4 \).

At least 4\( f \) face-edge adjacencies, and at most 2\( e \).
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At least $4f$ face-edge adjacencies,
  and at most $2e$.

.... $4f \leq 2e$
$K_{3,3}$ non-planarity.

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$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

Planar? No.

No cycles that are triangles.
  Cycles of length $\geq 4$.
At least $4f$ face-edge adjacencies,
  and at most $2e$.
  $\implies 4f \leq 2e$ for
$K_{3,3}$ non-planarity.

Euler: $v + \frac{2}{3} e \geq e + 2 \implies e \leq 3v - 6$

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

Planar? No.

No cycles that are triangles.
   Cycles of length $\geq 4$.
At least $4f$ face-edge adjacencies,
   and at most $2e$.
.... $4f \leq 2e$ for any
$K_{3,3}$ non-planarity.

\[ v + \frac{2}{3} e \geq e + 2 \implies e \leq 3v - 6 \]

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At least $4f$ face-edge adjacencies,

and at most $2e$.

\[ 4f \leq 2e \text{ for any bipartite} \]
$K_{3,3}$ non-planarity.

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.... $4f \leq 2e$ for any bipartite planar graph.
$K_{3,3}$ non-planarity.

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

Planar? No.

No cycles that are triangles.

Cycles of length $\geq 4$.

At least 4$f$ face-edge adjacencies,

and at most 2$e$.

.... 4$f$ $\leq$ 2$e$ for any bipartite planar graph.

Euler: $v + \frac{1}{2}e \geq e + 2$
$K_{3,3}$ non-planarity.

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!
Planar? No.

No cycles that are triangles.
  Cycles of length $\geq 4$.
At least 4$f$ face-edge adjacencies, 
  and at most 2$e$.
.... $4f \leq 2e$ for any bipartite planar graph.
Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$ for bipartite planar graph
$K_{3,3}$ non-planarity.

Euler: $v + \frac{2}{3} e \geq e + 2 \implies e \leq 3v - 6$

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

Planar? No.

No cycles that are triangles.

- Cycles of length $\geq 4$.
- At least $4f$ face-edge adjacencies, and at most $2e$.

$\implies 4f \leq 2e$ for any bipartite planar graph.

Euler: $v + \frac{1}{2} e \geq e + 2 \implies e \leq 2v - 4$ for bipartite planar graph.
$K_{3,3}$ non-planarity.

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

$K_{3,3}$? Edges? 9. Vertices. 6. $9 \leq 3(6) - 6$? Sure!

Planar? No.

No cycles that are triangles.
  Cycles of length $\geq 4$.
At least $4f$ face-edge adjacencies,
  and at most $2e$.

.... $4f \leq 2e$ for any bipartite planar graph.
Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$ for bipartite planar graph

$9 \not\leq 2(6) - 4$. 
**$K_{3,3}$ non-planarity.**

Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

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  Cycles of length $\geq 4$.

At least $4f$ face-edge adjacencies,
  and at most $2e$.

.... $4f \leq 2e$ for any bipartite planar graph.

Euler: $v + \frac{1}{2}e \geq e + 2 \implies e \leq 2v - 4$ for bipartite planar graph

$9 \not\leq 2(6) - 4. \implies K_{3,3}$ is not planar!
A tree is a connected acyclic graph.
A tree is a connected acyclic graph.
To tree or not to tree!
A tree is a connected acyclic graph.

To tree or not to tree!

![Diagram of trees and connected components]
A tree is a connected acyclic graph.

To tree or not to tree!

Yes.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No.
A tree is a connected acyclic graph.
To tree or not to tree!

Yes. No. Yes. No. No.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces?
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Recall: $e = v - 1$ for tree.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Recall: \( e = v - 1 \) for tree.
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Recall: \( e = v - 1 \) for tree.

One face for trees!
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Recall: \( e = v - 1 \) for tree.

One face for trees!

Euler works for trees: \( v + f = e + 2 \).
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Recall: $e = v - 1$ for tree.

One face for trees!

Euler works for trees: $v + f = e + 2$.

$v + 1 = v - 1 + 2$
A tree is a connected acyclic graph.

To tree or not to tree!

Yes. No. Yes. No. No.

Faces? 1. 2. 1. 1. 2.

Vertices/Edges. Recall: $e = v - 1$ for tree.

One face for trees!

Euler works for trees: $v + f = e + 2$.

$v + 1 = v - 1 + 2$
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$. 
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch:
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch: Induction on $e$. 
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch: Induction on $e$.

Base:
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

**Proof sketch:** Induction on \( e \).
Base: \( e = 0 \),
Euler's formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

**Proof sketch:** Induction on \( e \).
Base: \( e = 0, \; v = f = 1 \).
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch: Induction on $e$.
Base: $e = 0$, $v = f = 1$.
Induction Step:
Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$.

Base: $e = 0$, $v = f = 1$.

Induction Step:
- If it is a tree.
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$.
Base: $e = 0$, $v = f = 1$.
Induction Step:
  If it is a tree. Done.
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

**Proof sketch:** Induction on \( e \).
Base: \( e = 0, \ v = f = 1 \).
Induction Step:
  - If it is a tree. Done.
  - If not a tree.
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

**Proof sketch:** Induction on \( e \).

Base: \( e = 0 \), \( v = f = 1 \).

Induction Step:
- If it is a tree. Done.
- If not a tree.
  - Find a cycle.
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$.
Base: $e = 0, \ v = f = 1$.
Induction Step:
  - If it is a tree. Done.
  - If not a tree.
    Find a cycle. Remove edge.
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$.
Base: $e = 0$, $v = f = 1$.
Induction Step:
- If it is a tree. Done.
- If not a tree.
  - Find a cycle. Remove edge.

```
  f1
 /|
/  |
```

Outer face.

Joins two faces.
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$.
Base: $e = 0$, $v = f = 1$.
Induction Step:
  - If it is a tree. Done.
  - If not a tree.
    - Find a cycle. Remove edge.
    - Joins two faces.
    - New graph: $v$-vertices.
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof sketch: Induction on $e$.
Base: $e = 0$, $v = f = 1$.
Induction Step:
  If it is a tree. Done.
  If not a tree.
Find a cycle. Remove edge.

\[ f_{1} \]

Outer face.

Joins two faces.
New graph: $v$-vertices. $e - 1$ edges.
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

**Proof sketch:** Induction on \( e \).

Base: \( e = 0, \ v = f = 1 \).

Induction Step:
- If it is a tree. Done.
- If not a tree.
  
  Find a cycle. Remove edge.

\[
\text{Joins two faces. New graph: } v\text{-vertices. } e-1\text{ edges. } f-1\text{ faces.}
\]
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$.
Base: $e = 0, \ v = f = 1$.
Induction Step:
  If it is a tree. Done.
  If not a tree.
    Find a cycle. Remove edge.
    Joins two faces.
    New graph: $v$-vertices. $e - 1$ edges. $f - 1$ faces. Planar.
Euler’s formula.

Euler: Connected planar graph has $v + f = e + 2$.

**Proof sketch:** Induction on $e$.
Base: $e = 0$, $v = f = 1$.
Induction Step:
  - If it is a tree. Done.
  - If not a tree.
    - Find a cycle. Remove edge.
      
      ![Diagram of a cycle](image)

      Outer face.

    - Joins two faces.
    - $v + (f - 1) = (e - 1) + 2$ by induction hypothesis.
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

Proof sketch: Induction on \( e \).
Base: \( e = 0, \ v = f = 1 \).
Induction Step:
   If it is a tree. Done.
   If not a tree.
      Find a cycle. Remove edge.

\[
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\end{array}
\]

Outer face.

Joins two faces.
New graph: \( v \)-vertices. \( e - 1 \) edges. \( f - 1 \) faces. Planar.
\( v + (f - 1) = (e - 1) + 2 \) by induction hypothesis.
Therefore \( v + f = e + 2 \).
Euler’s formula.

Euler: Connected planar graph has \( v + f = e + 2 \).

**Proof sketch:** Induction on \( e \).

Base: \( e = 0, v = f = 1 \).

Induction Step:

- If it is a tree. Done.
- If not a tree.

  Find a cycle. Remove edge.

  ![Diagram of a cycle](image)

  Joins two faces.

  New graph: \( v \)-vertices. \( e - 1 \) edges. \( f - 1 \) faces. Planar.

\[
 v + (f - 1) = (e - 1) + 2 \text{ by induction hypothesis.}
\]

Therefore \( v + f = e + 2 \).
Summary

Graphs, trees, complete graphs, planar graphs.
Summary

Graphs, trees, complete graphs, planar graphs.
Euler’s formula.
Summary

Graphs, trees, complete graphs, planar graphs.
Euler’s formula.
Have a nice weekend!